Answer Key

1969 BC

1. C
2. E
3. B
4. D
5. E
6. B
7. D
8. C
9. D
10. A
11. B
12. E
13. C
14. D
15. B
16. B
17. B
18. E
19. C
20. A
21. B
22. E
23. D
24. C
25. A
26. C
27. C
28. D
29. C
30. D
31. C
32. B
33. A
34. D
35. A
36. B
37. D
38. A
39. D
40. E
41. D
42. B
43. E
44. E
45. E

1973 BC

1. A
2. D
3. A
4. C
5. B
6. D
7. D
8. B
9. A
10. A
11. E
12. D
13. D
14. A
15. C
16. A
17. C
18. D
19. D
20. E
21. B
22. C
23. C
24. A
25. B
26. D
27. E
28. C
29. A
30. B
31. E
32. C
33. A
34. C
35. C
36. E
37. E
38. B
39. D
40. C
41. D
42. D
43. E
44. A
45. E

1985 BC

1. D
2. A
3. B
4. D
5. D
6. E
7. A
8. C
9. B
10. A
11. A
12. A
13. B
14. C
15. C
16. C
17. B
18. C
19. D
20. C
21. B
22. A
23. C
24. D
25. C
26. E
27. E
28. E
29. D
30. B
31. D
32. E
33. C
34. A
35. B
36. E
37. A
38. C
39. A
40. A
41. C
42. E
43. E
44. A
45. D

1988 BC

1. A
2. D
3. B
4. E
5. C
6. C
7. A
8. A
9. D
10. D
11. A
12. B
13. B
14. A
15. E
16. A
17. D
18. E
19. B
20. E
21. D
22. E
23. E
24. D
25. D
26. C
27. B
28. E
29. B
30. C
31. C
32. E
33. E
34. C
35. A
36. E or D
37. D
38. C
39. C
40. E
41. B
42. A
43. A
44. A
45. B

1993 BC

1. A
2. C
3. E
4. B
5. D
6. A
7. A
8. B
9. D
10. E
11. E
12. E
13. C
14. B
15. D
16. A
17. A
18. B
19. B
20. E
21. A
22. B
23. D
24. C
25. D
26. B
27. C
28. A
29. E
30. C
31. A
32. B
33. A
34. E
35. A
36. E
37. B
38. C
39. C
40. C
41. C
42. E
43. A
44. E
45. D

1997 BC

1. C
2. E
3. A
4. C
5. C
6. A
7. C
8. E
9. A
10. B
11. C
12. A
13. B
14. C
15. D
16. B
17. B
18. C
19. D
20. E
21. A
22. C
23. E
24. D
25. A
76. D
77. E
78. A
79. D
80. B
81. D
82. B
83. E
84. C
85. D
86. A
87. B
88. C
89. D
90. B

1998 BC

1. C
2. A
3. D
4. A
5. A
6. E
7. E
8. B
9. D
10. E
11. A
12. E
13. B
14. E
15. B
16. C
17. D
18. B
19. D
20. E
21. C
22. A
23. E
24. C
25. C
26. E
27. D
28. C
76. D
77. E
78. B
79. A
80. B
81. B
82. B
83. C
84. B
85. C
86. C
87. D
88. C
89. A
90. A
91. E
92. D

1. C For horizontal asymptotes consider the limit as $x \to \pm\infty$: $t \to 0 \Rightarrow y = 0$ is an asymptote
For vertical asymptotes consider the limit as $y \to \pm\infty$: $t \to -1 \Rightarrow x = -1$ is an asymptote

2. E $y = (x+1)\tan^{-1} x$, $y' = \dfrac{x+1}{1+x^2} + \tan^{-1} x$

$$y'' = \frac{\left(1+x^2\right)(1) - (x+1)(2x)}{\left(1+x^2\right)^2} + \frac{1}{1+x^2} = \frac{2-2x}{\left(1+x^2\right)^2}$$

y'' changes sign at $x = 1$ only. The point of inflection is $\left(1, \pi/2\right)$

3. B $y = \sqrt{x}$, $y' = \dfrac{1}{2\sqrt{x}}$. By the Mean Value Theorem we have $\dfrac{1}{2\sqrt{c}} = \dfrac{2}{4} \Rightarrow c = 1$.

The point is $(1,1)$.

4. D $\displaystyle\int_0^8 \frac{dx}{\sqrt{1+x}}\,dx = 2\sqrt{1+x}\Big|_0^8 = 2(3-1) = 4$

5. E Using implicit differentiation, $6x + 2xy' + 2y + 2y \cdot y' = 0$. Therefore $y' = \dfrac{-2y-6x}{2x+2y}$.

When $x = 1$, $3 + 2y + y^2 = 2 \Rightarrow 0 = y^2 + 2y + 1 = (y+1)^2 \Rightarrow y = -1$

Therefore $2x + 2y = 0$ and so $\dfrac{dy}{dx}$ is not defined at $x = 1$.

6. B This is the derivative of $f(x) = 8x^8$ at $x = \dfrac{1}{2}$.

$$f'\left(\frac{1}{2}\right) = 64\left(\frac{1}{2}\right)^7 = \frac{1}{2}$$

7. D With $f(x) = x + \dfrac{k}{x}$, we need $0 = f'(-2) = 1 - \dfrac{k}{4}$ and so $k = 4$. Since $f''(-2) < 0$ for $k = 4$, f does have a relative maximum at $x = -2$.

8. C $h'(x) = 2f(x)\cdot f'(x) - 2g(x)\cdot g'(x) = 2f(x)\cdot\left(-g(x)\right) - 2g(x)\cdot f(x) = -4f(x)\cdot g(x)$

9. D $A=\frac{1}{2}\int_0^{2\pi}\left(\sqrt{3+\cos\theta}\right)^2 d\theta = 2\cdot\frac{1}{2}\int_0^{\pi}\left(\sqrt{3+\cos\theta}\right)^2 d\theta = \int_0^{\pi}(3+\cos\theta)\,d\theta$

10. A $\int_0^1 \frac{x^2}{x^2+1}dx = \int_0^1 \frac{x^2+1-1}{x^2+1}dx = \int_0^1\left(\frac{x^2+1}{x^2+1}-\frac{1}{x^2+1}\right)dx = \left(x-\tan^{-1}x\right)\Big|_0^1 = 1-\frac{\pi}{4}=\frac{4-\pi}{4}$

11. B Let L be the distance from $\left(x,-\frac{x^2}{2}\right)$ and $\left(0,-\frac{1}{2}\right)$.

$$L^2=(x-0)^2+\left(\frac{x^2}{2}-\frac{1}{2}\right)^2$$

$$2L\cdot\frac{dL}{dx}=2x+2\left(\frac{x^2}{2}-\frac{1}{2}\right)(x)$$

$$\frac{dL}{dx}=\frac{2x+2\left(\frac{x^2}{2}-\frac{1}{2}\right)(x)}{2L}=\frac{2x+x^3-x}{2L}=\frac{x^3+x}{2L}=\frac{x\left(x^2+1\right)}{2L}$$

$\frac{dL}{dx}<0$ for all $x<0$ and $\frac{dL}{dx}>0$ for all $x>0$, so the minimum distance occurs at $x=0$.

The nearest point is the origin.

12. E By the Fundamental Theorem of Calculus, if $F(x)=\int_0^x e^{-t^2}dt$ then $F'(x)=e^{-x^2}$.

13. C $\int_{-\pi/2}^{k}\cos x\,dx = 3\int_k^{\pi/2}\cos x\,dx$; $\sin k-\sin\left(-\frac{\pi}{2}\right)=3\left(\sin\frac{\pi}{2}-\sin k\right)$

$\sin k+1=3-3\sin k$; $4\sin k=2 \Rightarrow k=\pi/6$

14. D $y=x^2+2$ and $u=2x-1$, $\frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du}=(2x)\left(\frac{1}{2}\right)=x$

15. B The graphs do not need to intersect (eg. $f(x)=-e^{-x}$ and $g(x)=e^{-x}$). The graphs could intersect (e.g. $f(x)=2x$ and $g(x)=x$). However, if they do intersect, they will intersect no more than once because $f(x)$ grows faster than $g(x)$.

16. B $y' > 0 \Rightarrow y$ is increasing; $y'' < 0 \Rightarrow$ the graph is concave down. Only B meets these conditions.

17. B $y' = 20x^3 - 5x^4$, $y'' = 60x^2 - 20x^3 = 20x^2(3 - x)$. The only sign change in y'' is at $x = 3$. The only point of inflection is $(3,162)$.

18. E There is no derivative at the vertex which is located at $x = 3$.

19. C $\frac{dv}{dt} = \frac{1 - \ln t}{t^2} > 0$ for $0 < t < e$ and $\frac{dv}{dt} < 0$ for $t > e$, thus v has its maximum at $t = e$.

20. A $y(0) = 0$ and $y'(0) = \left.\frac{1/2}{\sqrt{1 - \frac{x^2}{4}}}\right|_{x=0} = \left.\frac{1}{\sqrt{4 - x^2}}\right|_{x=0} = \frac{1}{2}$. The tangent line is

$y = \frac{1}{2}x \Rightarrow x - 2y = 0$.

21. B $f'(x) = 2x - 2e^{-2x}$, $f'(0) = -2$, so f is decreasing

22. E $f(x) = \int_0^x \frac{1}{\sqrt{t^3 + 2}}\,dt$, $f(-1) = \int_0^{-1} \frac{1}{\sqrt{t^3 + 2}}\,dt = -\int_{-1}^{0} \frac{1}{\sqrt{t^3 + 2}}\,dt < 0$

$f(-1) < 0$ so E is false.

23. D $\frac{dy}{dx} = \frac{-xe^{-x^2}}{y} \Rightarrow 2y\,dy = -2xe^{-x^2}\,dx \Rightarrow y^2 = e^{-x^2} + C$

$4 = 1 + C \Rightarrow C = 3$; $y^2 = e^{-x^2} + 3 \Rightarrow y = \sqrt{e^{-x^2} + 3}$

24. C $y = \ln \sin x$, $y' = \frac{\cos x}{\sin x} = \cot x$

25. A $\int_m^{2m} \frac{1}{x}\,dx = \ln x \Big|_m^{2m} = \ln(2m) - \ln(m) = \ln 2$ so the area is independent of m.

26. C $\int_0^1 \sqrt{x^2-2x+1}\,dx = \int_0^1 |x-1|\,dx = \int_0^1 -(x-1)\,dx = -\frac{1}{2}(x-1)^2\Big|_0^1 = \frac{1}{2}$

Alternatively, the graph of the region is a right triangle with vertices at (0,0), (0,1), and (1,0). The area is $\frac{1}{2}$.

27. C $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$

28. D Use L'Hôpital's Rule: $\lim\limits_{x\to 0} \frac{e^{2x}-1}{\tan x} = \lim\limits_{x\to 0} \frac{2e^{2x}}{\sec^2 x} = 2$

29. C Make the subsitution $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\,d\theta$.

$$\int_0^1 \left(4-x^2\right)^{-3/2} dx = \int_0^{\pi/6} \frac{2\cos\theta}{8\cos^3\theta}\,d\theta = \frac{1}{4}\int_0^{\pi/6} \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta\Big|_0^{\pi/6} = \frac{1}{4}\cdot\frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{12}$$

30. D Substitute $-x$ for x in $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ to get $\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x}$

31. C $\frac{dy}{dx} = -y \Rightarrow y = ce^{-x}$ and $1 = ce^{-1} \Rightarrow c = e$; $y = e\cdot e^{-x} = e^{1-x}$

32. B $1 + 2^x + 3^x + 4^x + \cdots + n^x + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ where $p = -x$. This is a p-series and is convergent if $p > 1 \Rightarrow -x > 1 \Rightarrow x < -1$.

33. A $\frac{1}{3}\int_{-1}^{2} 3t^3 - t^2\,dt = \frac{1}{3}\left(\frac{3}{4}t^4 - \frac{1}{3}t^3\right)\Big|_{-1}^{2} = \frac{1}{3}\left(\left(12-\frac{8}{3}\right)-\left(\frac{3}{4}+\frac{1}{3}\right)\right) = \frac{11}{4}$

34. D $y' = -\frac{1}{x^2}$, so the desired curve satisfies $y' = x^2 \Rightarrow y = \frac{1}{3}x^3 + C$

35. A $a(t) = 24t^2$, $v(t) = 8t^3 + C$ and $v(0) = 0 \Rightarrow C = 0$. The particle is always moving to the right, so distance = $\int_0^2 8t^3 dt = 2t^4\Big|_0^2 = 32$.

36. B $y=\sqrt{4+\sin x}$, $y(0)=2$, $y'(0)=\dfrac{\cos 0}{2\sqrt{4+\sin 0}}=\dfrac{1}{4}$. The linear approximation to y is $L(x)=2+\dfrac{1}{4}x$. $L(1.2)=2+\dfrac{1}{4}(1.2)=2.03$

37. D This item uses the formal definition of a limit and is no longer part of the AP Course Description. Need to have $\left|(1-3x)-(-5)\right|<\varepsilon$ whenever $0<\left|x-2\right|<\delta$.
$\left|(1-3x)-(-5)\right|=\left|6-3x\right|=3\left|x-2\right|<\varepsilon$ if $\left|x-2\right|<\varepsilon/3$.
Thus we can use any $\delta<\varepsilon/3$. Of the five choices, the largest satisfying this condition is $\delta=\varepsilon/4$.

38. A Note $f(1)=\dfrac{1}{2}$. Take the natural logarithm of each side of the equation and then differentiate.

$$\ln f(x)=(2-3x)\ln\left(x^2+1\right);\ \frac{f'(x)}{f(x)}=(2-3x)\cdot\frac{2x}{x^2+1}-3\ln\left(x^2+1\right)$$

$$f'(1)=f(1)\left((-1)\cdot\frac{2}{2}-3\ln(2)\right)\Rightarrow f'(1)=\frac{1}{2}\left(-1-3\ln 2\right)=-\frac{1}{2}\left(\ln e+\ln 2^3\right)=-\frac{1}{2}\ln 8e$$

39. D $x=e\Rightarrow v=1,\ u=0,\ y=0;\ \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}=\left(\sec^2 u\right)\left(1+\dfrac{1}{v^2}\right)\left(\dfrac{1}{x}\right)=(1)(2)\left(e^{-1}\right)=\dfrac{2}{e}$

40. E One solution technique is to evaluate each integral and note that the value is $\dfrac{1}{n+1}$ for each.

Another technique is to use the substitution $u=1-x$; $\int_0^1(1-x)^n\,dx=\int_1^0 u^n\,(-du)=\int_0^1 u^n\,du$.

Integrals do not depend on the variable that is used and so $\int_0^1 u^n\,du$ is the same as $\int_0^1 x^n\,dx$.

41. D $\int_{-1}^{3} f(x)\,dx=\int_{-1}^{2}\left(8-x^2\right)dx+\int_2^3 x^2\,dx=\left(8x-\dfrac{1}{3}x^3\right)\Big|_{-1}^{2}+\dfrac{1}{3}x^3\Big|_2^3=27\tfrac{1}{3}$

42. B Use the technique of antiderivatives by parts to evaluate $\int x^2 \cos x\, dx$

$u = x^2 \qquad dv = \cos x\, dx$

$du = 2x\, dx \qquad v = \sin x$

$f(x) - \int 2x \sin x\, dx = \int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx + C$

$f(x) = x^2 \sin x + C$

43. E $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + \left(\sec^2 x\right)^2}\, dx = \int_a^b \sqrt{1 + \sec^4 x}\, dx$

44. E $y'' - y' - 2y = 0,\ y'(0) = -2,\ y(0) = 2$; the characteristic equation is $r^2 - r - 2 = 0$.

The solutions are $r = -1,\ r = 2$ so the general solution to the differential equation is

$y = c_1 e^{-x} + c_2 e^{2x}$ with $y' = -c_1 e^{-x} + 2c_2 e^{2x}$. Using the initial conditions we have the system:

$2 = c_1 + c_2$ and $-2 = -c_1 + 2c_2 \Rightarrow c_2 = 0,\ c_1 = 2$. The solution is $f(x) = 2e^{-x} \Rightarrow f(1) = 2e^{-1}$.

45. E The ratio test shows that the series is convergent for any value of x that makes $|x+1| < 1$. The solutions to $|x+1| = 1$ are the endpoints of the interval of convergence. Test $x = -2$ and $x = 0$ in the series. The resulting series are $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ which are both convergent.

The interval is $-2 \le x \le 0$.

1. A $f'(x) = e^{\frac{1}{x}} \cdot \dfrac{d\left(\frac{1}{x}\right)}{dx} = e^{\frac{1}{x}}\left(-\dfrac{1}{x^2}\right) = -\dfrac{e^{\frac{1}{x}}}{x^2}$

2. D $\int_0^3 (x+1)^{\frac{1}{2}}\,dx = \dfrac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^3 = \dfrac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \dfrac{2}{3}(8-1) = \dfrac{14}{3}$

3. A $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{(x+1)(x-1)}{x^2}$. $f'(x) > 0$ for $x < -1$ and for $x > 1$.

f is increasing for $x \le -1$ and for $x \ge 1$.

4. C The slopes will be negative reciprocals at the point of intersection.

$3x^2 = 3 \Rightarrow x = \pm 1$ and $x \ge 0$, thus $x = 1$ and the y values must be the same at $x = 1$.

$-\dfrac{1}{3} + b = 1 \Rightarrow b = \dfrac{4}{3}$

5. B $\int_{-1}^{2} \dfrac{|x|}{x}\,dx = \int_{-1}^{0} -1\,dx + \int_0^2 dx = -1 + 2 = 1$

6. D $f'(x) = \dfrac{(1)(x+1) - (x-1)(1)}{(x+1)^2}$, $f'(1) = \dfrac{2}{4} = \dfrac{1}{2}$

7. D $\dfrac{dy}{dx} = \dfrac{2x + 2y \cdot \dfrac{dy}{dx}}{x^2 + y^2}$ at $(1,0) \Rightarrow y' = \dfrac{2}{1} = 2$

8. B $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$, $y''' = -\cos x$, $y^{(4)} = \sin x$

9. A $y' = 2\cos 3x \cdot \dfrac{d}{dx}(\cos 3x) = 2\cos 3x \cdot (-\sin 3x) \cdot \dfrac{d}{dx}(3x) = 2\cos 3x \cdot (-\sin 3x) \cdot 3$

$y' = -6 \sin 3x \cos 3x$

10. A $L=\int_0^b \sqrt{1+(y')^2}\,dx=\int_0^b \sqrt{1+\left(\frac{\sec x\tan x}{\sec x}\right)^2}\,dx$

$=\int_0^b \sqrt{1+(\tan x)^2}\,dx=\int_0^b \sqrt{\sec^2 x}\,dx=\int_0^b \sec x\,dx$

11. E $dy=\left(x\cdot\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}(2x)+\left(1+x^2\right)^{\frac{1}{2}}\right)dx$; $dy=(0+1)(2)=2$

12. D $\frac{1}{n}=\int_1^k x^{n-1}\,dx=\left.\frac{x^n}{n}\right|_1^k \Rightarrow \frac{1}{n}=\frac{k^n}{n}-\frac{1}{n};\ \frac{k^n}{n}=\frac{2}{n}\Rightarrow k=2^{\frac{1}{n}}$

13. D $v(t)=8t-3t^2+C$ and $v(1)=25\Rightarrow C=20$ so $v(t)=8t-3t^2+20$.

$s(4)-s(2)=\int_2^4 v(t)\,dt=\left.\left(4t^2-t^3+20t\right)\right|_2^4=32$

14. A $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2e^t}{2t}=\frac{e^t}{t}$

15. C Area $=\int_0^2 e^{\frac{1}{2}x}\,dx=\left.2e^{\frac{1}{2}x}\right|_0^2=2(e-1)$

16. A $\sin t=t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots\Rightarrow\frac{\sin t}{t}=1-\frac{t^2}{3!}+\frac{t^4}{5!}-\frac{t^6}{7!}+\cdots$

17. C $\frac{dN}{dt}=3000e^{\frac{2}{5}t}$, $N=7500e^{\frac{2}{5}t}+C$ and $N(0)=7500\Rightarrow C=0$

$N=7500e^{\frac{2}{5}t}$, $N(5)=7500e^2$

18. D D could be false, consider $g(x)=1-x$ on $[0,1]$. A is true by the Extreme Value Theorem, B is true because g is a function, C is true by the Intermediate Value Theorem, and E is true because g is continuous.

19. D I is a convergent p-series, $p = 2 > 1$
II is the Harmonic series and is known to be divergent,
III is convergent by the Alternating Series Test.

20. E $\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int \left(4-x^2\right)^{\frac{1}{2}}(-2x\,dx) = -\frac{1}{2}\cdot\frac{2}{3}\left(4-x^2\right)^{\frac{3}{2}} + C = -\frac{1}{3}\left(4-x^2\right)^{\frac{3}{2}} + C$

21. B $\int_0^1 (x+1)e^{x^2+2x}\,dx = \frac{1}{2}\int_0^1 e^{x^2+2x}\left((2x+2)\,dx\right) = \frac{1}{2}\left(e^{x^2+2x}\right)\Big|_0^1 = \frac{1}{2}\left(e^3 - e^0\right) = \frac{e^3-1}{2}$

22. C $x'(t) = t+1 \Rightarrow x(t) = \frac{1}{2}(t+1)^2 + C$ and $x(0) = 1 \Rightarrow C = \frac{1}{2} \Rightarrow x(t) = \frac{1}{2}(t+1)^2 + \frac{1}{2}$

$x(1) = \frac{5}{2},\ y(1) = \ln\frac{5}{2};\qquad \left(\frac{5}{2}, \ln\frac{5}{2}\right)$

23. C $\lim_{h\to 0}\frac{\ln(2+h)-\ln 2}{h} = f'(2)$ where $f(x) = \ln x$; $f'(x) = \frac{1}{x} \Rightarrow f'(2) = \frac{1}{2}$

24. A This item uses the formal definition of a limit and is no longer part of the AP Course Description. $\left|f(x)-7\right| = \left|(3x+1)-7\right| = \left|3x-6\right| = 3\left|x-2\right| < \varepsilon$ whenever $\left|x-2\right| < \frac{\varepsilon}{3}$.

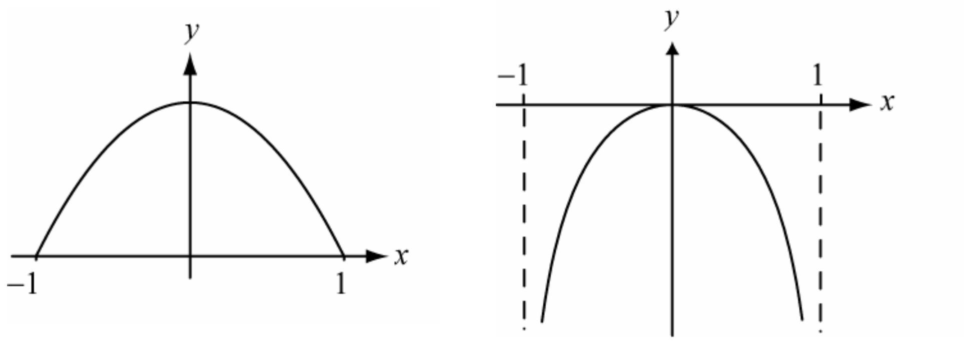
Any $\delta < \frac{\varepsilon}{3}$ will be sufficient and $\frac{\varepsilon}{4} < \frac{\varepsilon}{3}$, thus the answer is $\frac{\varepsilon}{4}$.

25. B $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \int_0^{\frac{\pi}{4}}\left(\sec^2 x - 1\right)dx = \left(\tan x - x\right)\Big|_0^{\pi/4} = 1 - \frac{\pi}{4}$

26. D For x in the interval (–1, 1), $g(x) = \left|x^2-1\right| = -(x^2-1)$ and so $y = \ln g(x) = \ln(-(x^2-1))$.
Therefore

$$y' = \frac{2x}{x^2-1},\quad y'' = \frac{\left(x^2-1\right)(2)-(2x)(2x)}{\left(x^2-1\right)^2} = \frac{-2x^2-2}{\left(x^2-1\right)^2} < 0$$

Alternative graphical solution: Consider the graphs of $g(x)=\left|x^2-1\right|$ and $\ln g(x)$.



$g(x)=\left|x^2-1\right|$ $\ln\left|x^2-1\right|$

concave down

27. E $f'(x)=x^2-8x+12=(x-2)(x-6)$; the candidates are: $x=0,2,6,9$

x	0	2	6	9
$f(x)$	−5	$17/3$	−5	22

the maximum is at $x=9$

28. C $x=\sin^2 y \Rightarrow dx=2\sin y\cos y\,dy$; when $x=0,\ y=0$ and when $x=\frac{1}{2},\ y=\frac{\pi}{4}$

$$\int_0^{\frac{1}{2}}\frac{\sqrt{x}}{\sqrt{1-x}}dx=\int_0^{\frac{\pi}{4}}\frac{\sin y}{\sqrt{1-\sin^2 y}}\cdot 2\sin y\cos y\,dy=\int_0^{\frac{\pi}{4}}2\sin^2 y\,dy$$

29. A Let $z=y'$. Then $z=e$ when $x=0$. Thus $y''=2y'\Rightarrow z'=2z$. Solve this differential equation.

$z=Ce^{2x}$; $e=Ce^0\Rightarrow C=e\Rightarrow y'=z=e^{2x+1}$. Solve this differential equation.

$$y=\frac{1}{2}e^{2x+1}+K;\ e=\frac{1}{2}e^1+K\Rightarrow K=\frac{1}{2}e;\ y=\frac{1}{2}e^{2x+1}+\frac{1}{2}e,\ y(1)=\frac{1}{2}e^3+\frac{1}{2}e=\frac{1}{2}e\left(e^2+1\right)$$

Alternative Solution: $y''=2y'\Rightarrow y'=Ce^{2x}=e\cdot e^{2x}$. Therefore $y'(1)=e^3$.

$y'(1)-y'(0)=\int_0^1 y''(x)dx=\int_0^1 2y'(x)dx=2y(1)-2y(0)$ and so

$$y(1)=\frac{y'(1)-y'(0)+2y(0)}{2}=\frac{e^3+e}{2}.$$

30. B $\int_1^2 \frac{x-4}{x^2}\,dx = \int_1^2 \left(\frac{1}{x} - 4x^{-2}\right)dx = \left(\ln x + \frac{4}{x}\right)\Big|_1^2 = (\ln 2 + 2) - (\ln 1 + 4) = \ln 2 - 2$

31. E $f'(x) = \dfrac{\frac{d}{dx}(\ln x)}{\ln x} = \dfrac{\frac{1}{x}}{\ln x} = \dfrac{1}{x\ln x}$

32. C Take the log of each side of the equation and differentiate. $\ln y = \ln x^{\ln x} = \ln x \cdot \ln x = (\ln x)^2$

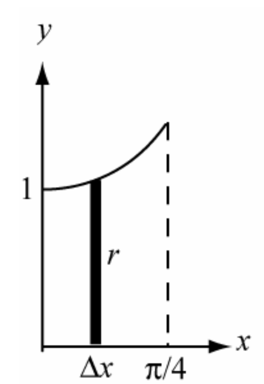
$$\frac{y'}{y} = 2\ln x \cdot \frac{d}{dx}(\ln x) = \frac{2}{x}\ln x \Rightarrow y' = x^{\ln x}\left(\frac{2}{x}\ln x\right)$$

33. A $f(-x) = -f(x) \Rightarrow f'(-x)\cdot(-1) = -f'(x) \Rightarrow f'(-x) = -f'(x)$ thus $f'(-x_0) = -f'(x_0)$.

34. C $\frac{1}{2}\int_0^2 \sqrt{x}\,dx = \frac{1}{2}\cdot\frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 = \frac{1}{3}\cdot 2^{\frac{3}{2}} = \frac{2}{3}\sqrt{2}$

35. C Washers: $\sum \pi r^2 \Delta x$ where $r = y = \sec x$.

$$\text{Volume} = \pi\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \pi \tan x\Big|_0^{\frac{\pi}{4}} = \pi\left(\tan\frac{\pi}{4} - \tan 0\right) = \pi$$



36. E $\int_0^1 \frac{x+1}{x^2+2x-3}\,dx = \frac{1}{2}\lim_{L\to 1^-}\int_0^L \frac{2x+2}{x^2+2x-3}\,dx = \frac{1}{2}\lim_{L\to 1^-} \ln\left|x^2+2x-3\right|\Big|_0^L$

$$= \frac{1}{2}\lim_{L\to 1^-}\left(\ln\left|L^2+2L-3\right| - \ln\left|-3\right|\right) = -\infty. \quad \text{Divergent}$$

37. E $\lim_{x\to 0}\frac{1-\cos^2 2x}{x^2} = \lim_{x\to 0}\frac{\sin^2 2x}{x^2} = \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{\sin 2x}{2x}\cdot 4 = 1\cdot 1\cdot 4 = 4$

38. B Let $z = x - c$. $5 = \int_1^2 f(x-c)\,dx = \int_{1-c}^{2-c} f(z)\,dz$

39. D $h'(x) = f'(g(x))\cdot g'(x)$; $h'(1) = f'(g(1))\cdot g'(1) = f'(2)\cdot g'(1) = (-4)(-3) = 12$

40. C Area $=\frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2 d\theta=\int_0^{\pi}\left(1-2\cos\theta+\cos^2\theta\right)d\theta$; $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$

$$\text{Area}=\int_0^{\pi}\left(1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right)d\theta=\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi}=\frac{3}{2}\pi$$

41. D $\int_{-1}^{1} f(x)\,dx=\int_{-1}^{0}(x+1)\,dx+\int_0^1\cos(\pi x)\,dx$

$$=\frac{1}{2}(x+1)^2\Big|_{-1}^{0}+\frac{1}{\pi}\sin(\pi x)\Big|_0^1=\frac{1}{2}+\frac{1}{\pi}(\sin\pi-\sin 0)=\frac{1}{2}$$

42. D $\Delta x=\frac{1}{3}$; $T=\frac{1}{2}\cdot\frac{1}{3}\left(1^2+2\left(\frac{4}{3}\right)^2+2\left(\frac{5}{3}\right)^2+2^2\right)=\frac{127}{54}$

43. E Use the technique of antiderivatives by part:

$u=\sin^{-1}x \qquad dv=dx$

$du=\frac{dx}{\sqrt{1-x^2}} \qquad v=x$

$$\int\sin^{-1}x\,dx=x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}\,dx$$

44. A Multiply both sides of $x=xf'(x)-f(x)$ by $\frac{1}{x^2}$. Then $\frac{1}{x}=\frac{xf'(x)-f(x)}{x^2}=\frac{d}{dx}\left(\frac{f(x)}{x}\right)$.

Thus we have $\frac{f(x)}{x}=\ln|x|+C\Rightarrow f(x)=x\left(\ln|x|+C\right)=x\left(\ln|x|-1\right)$ since $f(-1)=1$.

Therefore $f(e^{-1})=e^{-1}\left(\ln\left|e^{-1}\right|-1\right)=e^{-1}(-1-1)=-2e^{-1}$

This was most likely the solution students were expected to produce while solving this problem on the 1973 multiple-choice exam. However, the problem itself is not well-defined. A solution to an initial value problem should be a function that is differentiable on an interval containing the initial point. In this problem that would be the domain $x<0$ since the solution requires the choice of the branch of the logarithm function with $x<0$. Thus one cannot ask about the value of the function at $x=e^{-1}$.

45. E $F'(x)=xg'(x)$ with $x\ge 0$ and $g'(x)<0\Rightarrow F'(x)<0\Rightarrow F$ is not increasing.

1. D $\int_0^2 (4x^3+2)\,dx = (x^4+2x)\Big|_0^2 = (16+4)-(1+2) = 17$

2. A $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1) = 15x^2(x-1)(x+1)$, changes sign from positive to negative only at $x=-1$. So f has a relative maximum at $x=-1$ only.

3. B $\int_1^2 \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int_1^2 \frac{(2x+2)\,dx}{x^2+2x} = \frac{1}{2}\ln\left|x^2+2x\right|\Big|_1^2 = \frac{1}{2}(\ln 8 - \ln 3)$

4. D $x(t) = t^2 - 1 \Rightarrow \frac{dx}{dt} = 2t$ and $\frac{d^2x}{dt^2} = 2$; $y(t) = t^4 - 2t^3 \Rightarrow \frac{dy}{dt} = 4t^3 - 6t^2$ and $\frac{d^2y}{dt^2} = 12t^2 - 12t$

$a(t) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right) = (2,\, 12t^2 - 12t) \Rightarrow a(1) = (2, 0)$

5. D Area $= \int_{x_1}^{x_2} (\text{top curve} - \text{bottom curve})\,dx$, $x_1 < x_2$; Area $= \int_{-1}^{a} (f(x) - g(x))\,dx$

6. E $f(x) = \frac{x}{\tan x}$, $f'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$, $f'\left(\frac{\pi}{4}\right) = \frac{1 - \frac{\pi}{4}\cdot\left(\sqrt{2}\right)^2}{1} = 1 - \frac{\pi}{2}$

7. A $\int \frac{du}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) \Rightarrow \int \frac{dx}{\sqrt{25-x^2}}\,dx = \sin^{-1}\left(\frac{x}{5}\right) + C$

8. C $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = f'(2)$ so the derivative of f at $x=2$ is 0.

9. B Take the derivative of each side of the equation with respect to x.

$2xyy' + y^2 + 2xy' + 2y = 0$, substitute the point $(1,2)$

$(1)(4)y' + 2^2 + (2)(1)y' + (2)(2) = 0 \Rightarrow y = -\frac{4}{3}$

10. A Take the derivative of the general term with respect to x: $\sum_{n=1}^{\infty} (-1)^{n+1} x^{2n-2}$

11. A $\frac{d}{dx}\left(\ln\left(\frac{1}{1-x}\right)\right) = \frac{d}{dx}\left(-\ln(1-x)\right) = -\left(\frac{-1}{1-x}\right) = \frac{1}{1-x}$

12. A Use partial fractions to rewrite $\frac{1}{(x-1)(x+2)}$ as $\frac{1}{3}\left(\frac{1}{x-1}-\frac{1}{x+2}\right)$

$$\int \frac{1}{(x-1)(x+2)}dx=\frac{1}{3}\int\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx=\frac{1}{3}\left(\ln|x-1|-\ln|x+2|\right)+C=\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$$

13. B $f(0)=0,\ f(3)=0,\ f'(x)=3x^2-6x;$ by the Mean Value Theorem,
$f'(c)=\frac{f(3)-f(0)}{3}=0$ for $c\in(0,3)$.
So, $0=3c^2-6c=3c(c-2)$. The only value in the open interval is 2.

14. C I. convergent: p-series with $p=2>1$
II. divergent: Harmonic series which is known to diverge
III. convergent: Geometric with $|r|=\frac{1}{3}<1$

15. C $x(t)=4+\int_0^t(2w-4)\,dw=4+(w^2-4w)\Big|_0^t=4+t^2-4t=t^2-4t+4$

or, $x(t)=t^2-4t+C$, $x(0)=4\Rightarrow C=4$ so, $x(t)=t^2-4t+4$

16. C For $f(x)=x^{\frac{1}{3}}$ we have continuity at $x=0$, however, $f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$ is not defined at $x=0$.

17. B $f'(x)=(1)\cdot\ln(x^2)+x\cdot\frac{\frac{d}{dx}(x^2)}{x^2}=\ln(x^2)+\frac{2x^2}{x^2}=\ln(x^2)+2$

18. C $\int\sin(2x+3)\,dx=\frac{1}{2}\int\sin(2x+3)(2dx)=-\frac{1}{2}\cos(2x+3)+C$

19. D $g(x)=e^{f(x)},\ g'(x)=e^{f(x)}\cdot f'(x),\ g''(x)=e^{f(x)}\cdot f''(x)+f'(x)\cdot e^{f(x)}\cdot f'(x)$
$g''(x)=e^{f(x)}\left(f''(x)+\left(f'(x)^2\right)\right)=h(x)e^{f(x)}\Rightarrow h(x)=f''(x)+\left(f'(x)^2\right)$

20. C Look for concavity changes, there are 3.

21. B Use the technique of antiderivatives by parts:

$u = f(x) \qquad dv = \sin x\,dx$

$du = f'(x)\,dx \qquad v = -\cos x$

$\int f(x)\sin x\,dx = -f(x)\cos x + \int f'(x)\cos x\,dx$ and we are given that

$\int f(x)\sin x\,dx = -f(x)\cos x + \int 3x^2\cos x\,dx \Rightarrow f'(x) = 3x^2 \Rightarrow f(x) = x^3$

22. A $A = \pi r^2$, $A = 64\pi$ when $r = 8$. Take the derivative with respect to *t*.

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt};\ 96\pi = 2\pi(8)\cdot\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 6$

23. C $\lim_{h\to 0}\frac{\int_1^{1+h}\sqrt{x^5+8}\,dx}{h} = \lim_{h\to 0}\frac{F(1+h)-F(1)}{h} = F'(1)$ where $F'(x) = \sqrt{x^5+8}$. $F'(1)=3$

Alternate solution by L'Hôpital's Rule: $\lim_{h\to 0}\frac{\int_1^{1+h}\sqrt{x^5+8}\,dx}{h} = \lim_{h\to 0}\frac{\sqrt{(1+h)^5+8}}{1} = \sqrt{9} = 3$

24. D Area $= \frac{1}{2}\int_0^{\pi/2}\sin^2(2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{\pi}{8}$

25. C At rest when $v(t) = 0$. $v(t) = e^{-2t} - 2te^{-2t} = e^{-2t}(1-2t)$, $v(t) = 0$ at $t = \frac{1}{2}$ only.

26. E Apply the log function, simplify, and differentiate. $\ln y = \ln(\sin x)^x = x\ln(\sin x)$

$\frac{y'}{y} = \ln(\sin x) + x\cdot\frac{\cos x}{\sin x} \Rightarrow y' = y\left(\ln(\sin x) + x\cdot\cot x\right) = (\sin x)^x\left(\ln(\sin x) + x\cdot\cot x\right)$

27. E Each of the right-hand sides represent the area of a rectangle with base length $(b-a)$.

I. Area under the curve is less than the area of the rectangle with height $f(b)$.

II. Area under the curve is more than the area of the rectangle with height $f(a)$.

III. Area under the curve is the same as the area of the rectangle with height $f(c)$, $a < c < b$. Note that this is the Mean Value Theorem for Integrals.

28. E $\int e^{x+e^x}\,dx = \int e^{e^x}(e^x dx)$. This is of the form $\int e^u\,du,\ u = e^x$, so $\int e^{x+e^x}\,dx = e^{e^x} + C$

29. D Let $x - \frac{\pi}{4} = t$. $\lim_{x \to \frac{\pi}{4}} \frac{\sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{t \to 0} \frac{\sin t}{t} = 1$

30. B At t = 1, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left.\frac{\frac{3}{2\sqrt{3t+1}}}{3t^2 - 1}\right|_{t=1} = \frac{\frac{3}{4}}{3-1} = \frac{3}{8}$

31. D The center is $x = 1$, so only C, D, or E are possible. Check the endpoints.

At $x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test.

At $x = 2$: $\sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series and known to diverge.

32. E $y(-1) = -6$, $y'(-1) = 3x^2 + 6x + 7\Big|_{x=-1} = 4$, the slope of the normal is $-\frac{1}{4}$ and an equation for the normal is $y + 6 = -\frac{1}{4}(x+1) \Rightarrow x + 4y = -25$.

33. C This is the differential equation for exponential growth.

$y = y(0)e^{-2t} = e^{-2t}$; $\frac{1}{2} = e^{-2t}$; $-2t = \ln\left(\frac{1}{2}\right) \Rightarrow t = -\frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{1}{2}\ln 2$

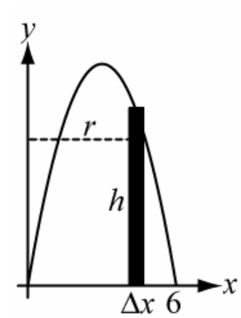
34. A This topic is no longer part of the AP Course Description. $\sum 2\pi\rho\,\Delta s$ where $\rho = x = y^3$

$$\text{Surface Area} = \int_0^1 2\pi y^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi y^3 \sqrt{1 + \left(3y^2\right)^2}\, dy = 2\pi \int_0^1 y^3 \sqrt{1 + 9y^4}\, dy$$

35. B Use shells (which is no longer part of the AP Course Description)

$\sum 2\pi rh\,\Delta x$ where $r = x$ and $h = y = 6x - x^2$

$$\text{Volume} = 2\pi \int_0^6 x\left(6x - x^2\right) dx$$



36. E $\int_{-1}^{1}\frac{3}{x^2}dx = 2\int_0^1 \frac{3}{x^2}dx = 2\lim_{L\to 0^+}\int_L^1 \frac{3}{x^2}dx = 2\lim_{L\to 0^+} -\frac{3}{x}\Big|_L^1$ which does not exist.

37. A This topic is no longer part of the AP Course Description. $y = y_h + y_p$ where $y_h = ce^{-x}$ is the solution to the homogeneous equation $\frac{dy}{dx} + y = 0$ and $y_p = \left(Ax^2 + Bx\right)e^{-x}$ is a particular solution to the given differential equation. Substitute y_p into the differential equation to determine the values of A and B. The answer is $A = \frac{1}{2},\ B = 0$.

38. C $\lim_{x\to\infty}\left(1+5e^x\right)^{1/x} = \lim_{x\to\infty} e^{\ln\left(1+5e^x\right)^{1/x}} = e^{\lim_{x\to\infty}\ln\left(1+5e^x\right)^{1/x}} = e^{\lim_{x\to\infty}\frac{\ln\left(1+5e^x\right)}{x}} = e^{\lim_{x\to\infty}\frac{5e^x}{1+5e^x}} = e$

39. A Square cross sections: $\sum y^2\,\Delta x$ where $y = e^{-x}$. $V = \int_0^3 e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_0^3 = \frac{1}{2}\left(1-e^{-6}\right)$

40. A $u = \frac{x}{2},\ du = \frac{1}{2}dx$; when $x = 2$, $u = 1$ and when $x = 4$, $u = 2$

$$\int_2^4 \frac{1-\left(\frac{x}{2}\right)^2}{x}dx = \int_1^2 \frac{1-u^2}{2u}\cdot 2\,du = \int_1^2 \frac{1-u^2}{u}\,du$$

41. C $y' = x^{\frac{1}{2}},\ L = \int_0^3 \sqrt{1+\left(y'\right)^2}\,dx = \int_0^3 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^3 = \frac{2}{3}\left(4^{3/2}-1^{3/2}\right) = \frac{2}{3}(8-1) = \frac{14}{3}$

42. E Since $e^u = 1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots$, then $e^{3x} = 1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\cdots$

The coefficient we want is $\frac{3^3}{3!} = \frac{9}{2}$

43. E Graphs A and B contradict $f'' < 0$. Graph C contradicts $f'(0)$ does not exist. Graph D contradicts continuity on the interval $[-2,3]$. Graph E meets all given conditions.

44. A $\frac{dy}{dx} = 3x^2y \Rightarrow \frac{dy}{y} = 3x^2dx \Rightarrow \ln|y| = x^3 + K;\ y = Ce^{x^3}$ and $y(0) = 8$ so, $y = 8e^{x^3}$

45. D The expression is a Riemann sum with $\Delta x = \frac{1}{n}$ and $f(x) = x^2$.

The evaluation points are: $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots, \frac{3n}{n}$

Thus the right Riemann sum is for $x = 0$ to $x = 3$. The limit is equal to $\int_0^3 x^2\,dx$.

1988 Calculus BC Solutions

1. A $\int_0^1 \left(x-x^2\right)dx = \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^1 = \frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ only.

2. D $\int_0^1 x\left(x^2+2\right)^2 dx = \frac{1}{2}\int_0^1 \left(x^2+2\right)^2\left(2x\,dx\right) = \frac{1}{2}\cdot\frac{1}{3}\left(x^2+2\right)^3\Big|_0^1 = \frac{1}{6}\left(3^3-2^3\right)=\frac{19}{6}$

3. B $f(x)=\ln\sqrt{x}=\frac{1}{2}\ln x;\ f'(x)=\frac{1}{2}\cdot\frac{1}{x}\Rightarrow f''(x)=-\frac{1}{2x^2}$

4. E $\left(\frac{uv}{w}\right)' = \frac{(uv'+u'v)w-uvw'}{w^2} = \frac{uv'w+u'vw-uvw'}{w^2}$

5. C $\lim_{x\to a} f(x)=f(a)$ for all values of a except 2.
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x-2)=0\neq -1=f(2)$

6. C $2y\cdot y'-2x\cdot y'-2y=0\Rightarrow y'=\frac{y}{y-x}$

7. A $\int_2^\infty \frac{dx}{x^2} = \lim_{L\to\infty}\int_2^L \frac{dx}{x^2} = \lim_{L\to\infty}\left(-\frac{1}{x}\right)\Big|_2^L = \lim_{L\to\infty}\left(\frac{1}{2}-\frac{1}{L}\right)=\frac{1}{2}$

8. A $f'(x)=e^x$, $f'(2)=e^2$, $\ln e^2=2$

9. D II does not work since the slope of f at $x=0$ is not equal to $f'(0)$. Both I and III could work. For example, $f(x)=e^x$ in I and $f(x)=\sin x$ in III.

10. D This limit is the derivative of $\sin x$.

11. A The slope of the line is $-\frac{1}{7}$, so the slope of the tangent line at $x=1$ is $7\Rightarrow f'(1)=7$.

12. B $v(t)=3t+C$ and $v(2)=10\Rightarrow C=4$ and $v(t)=3t+4$.
Distance $=\int_0^2\left(3t+4\right)dt=\frac{3}{2}x^2+4t\Big|_0^2=14$

13. B The Maclaurin series for $\sin t$ is $t-\dfrac{t^3}{3!}+\dfrac{t^5}{5!}-\cdots$. Let $t=2x$.

$$\sin(2x)=2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots+\frac{(-1)^{n-1}(2x)^{2n-1}}{(2n-1)!}+\cdots$$

14 A Use the Fundamental Theorem of Calculus: $\sqrt{1+(x^2)^3}\cdot\dfrac{d(x^2)}{dx}=2x\sqrt{1+x^6}$

15. E $x=t^2+1,\ \dfrac{dx}{dt}=2t,\ \dfrac{d^2x}{dt^2}=2;\ y=\ln(2t+3),\ \dfrac{dy}{dt}=\dfrac{2}{2t+3};\ \dfrac{d^2y}{dt^2}=-\dfrac{4}{(2t+3)^2}$

16. A Use the technique of antiderivatives by parts

$u=x \qquad dv=e^{2x}\,dx$

$du=dx \qquad v=\frac{1}{2}e^{2x}$

$$\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}\,dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$$

17. D Use partial fractions:

$$\int_2^3\frac{3}{(x-1)(x+1)}dx=\int_2^3\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx=\left(\ln|x-1|-\ln|x+2|\right)\Big|_2^3=\ln 2-\ln 5-\ln 1+\ln 4=\ln\frac{8}{5}$$

18. E $\Delta x=\dfrac{4-(-2)}{3}=2,\ T=\dfrac{1}{2}(2)\left(\dfrac{e^4}{2}+2\cdot\dfrac{e^2}{2}+2\cdot\dfrac{e^0}{2}+\dfrac{e^{-2}}{2}\right)=\dfrac{1}{2}\left(e^4+2e^2+2e^0+e^{-2}\right)$

19. B Make a sketch. $x<-2$ one zero, $-2<x<5$ no zeros, $x>5$ one zero for a total of 2 zeros

20. E This is the definition of a limit.

21. D $\dfrac{1}{2}\displaystyle\int_1^3\frac{1}{x}dx=\frac{1}{2}\ln x\Big|_1^3=\frac{1}{2}\left(\ln 3-\ln 1\right)=\frac{1}{2}\ln 3$

22. E Quick Solution: f' must have a factor of f which makes E the only option. Or,

$$\ln f(x)=x\ln(x^2+1)\Rightarrow\frac{f'(x)}{f(x)}=x\cdot\frac{2x}{x^2+1}+\ln(x^2+1)\Rightarrow f'(x)=f(x)\cdot\left(\frac{2x^2}{x^2+1}+\ln(x^2+1)\right)$$

23. E $r = 0$ when $\cos 3\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{6}$. The region is for the interval from $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$.

$$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (4\cos 3\theta)^2\, d\theta$$

24. D $f'(x) = 3x^2 - 4x$, $f(0) = 0$ and $f(2) = 0$. By the Mean Value Theorem,

$$0 = \frac{f(2) - f(0)}{2 - 0} = f'(c) = 3c^2 - 4c \text{ for } c \in (0, 2). \text{ So, } c = 4/3.$$

25. D Square cross-sections: $\sum y^2\,\Delta x$ where $y = 4x^2$. Volume $= \int_0^1 16x^4\, dx = \frac{16}{5}x^4\Big|_0^1 = \frac{16}{5}$.

26. C This is not true if f is not an even function.

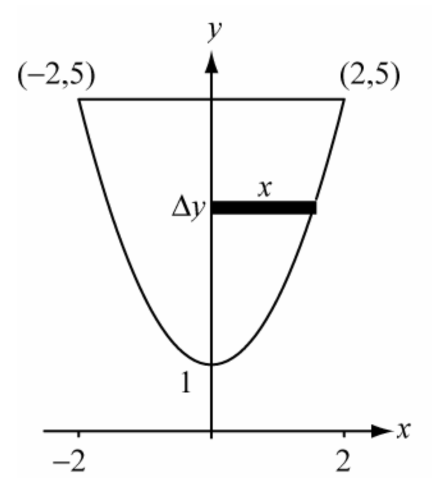
27. B $y'(x) = 3x^2 + 2ax + b$, $y''(x) = 6x + 2a$, $y''(1) = 0 \Rightarrow a = -3$
$y(1) = -6$ so, $-6 = 1 + a + b - 4 \Rightarrow -6 = 1 - 3 + b - 4 \Rightarrow b = 0$

28. E

$$\frac{\frac{d}{dx}\left(\cos\left(\frac{\pi}{x}\right)\right)}{\cos\left(\frac{\pi}{x}\right)} = \frac{-\sin\left(\frac{\pi}{x}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{x}\right)}{\cos\left(\frac{\pi}{x}\right)} = \frac{-\sin\left(\frac{\pi}{x}\right)\cdot\left(-\frac{\pi}{x^2}\right)}{\cos\left(\frac{\pi}{x}\right)} = \frac{\pi}{x^2}\tan\left(\frac{\pi}{x}\right)$$

29. B Disks: $\sum \pi x^2 \Delta y$ where $x^2 = y - 1$.

$$\text{Volume} = \pi\int_1^5 (y-1)\, dy = \frac{\pi}{2}(y-1)^2\Big|_1^5 = 8\pi$$



30. C This is an infinite geometric series with ratio $\frac{1}{3}$ and first term $\frac{1}{3^n}$.

$$\text{Sum} = \frac{\text{first}}{1-\text{ratio}} = \frac{\left(\frac{1}{3^n}\right)}{1-\frac{1}{3}} = \frac{3}{2}\cdot\left(\frac{1}{3^n}\right)$$

31. C This integral gives $\frac{1}{4}$ of the area of the circle with center at the origin and radius = 2.

$$\frac{1}{4}\left(\pi\cdot 2^2\right)=\pi$$

32. E No longer covered in the AP Course Description. The solution is of the form $y = y_h + y_p$ where y_h is the solution to $y' - y = 0$ and the form of y_p is $Ax^2 + Bx + K$. Hence $y_h = Ce^x$. Substitute y_p into the original differential equation to determine the values of A, B, and K.

Another technique is to substitute each of the options into the differential equation and pick the one that works. Only (A), (B), and (E) are viable options because of the form for y_h. Both (A) and (B) fail, so the solution is (E).

33. E $L = \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1+\left(3x^2\right)^2}\,dx = \int_0^2 \sqrt{1+9x^4}\,dx$

34. C At $t = 1$, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left.\frac{4t^3+4t}{3t^2+1}\right|_{t=1} = \frac{8}{4} = 2$; the point at $t=1$ is $(2,3)$. $y = 3+2\left(x-2\right) = 2x-1$

35. A Quick solution: For large x the exponential function dominates any polynomial, so $\lim\limits_{x\to+\infty}\frac{x^k}{e^x} = 0$.

Or, repeated use of L'Hôpital's rule gives $\Rightarrow \lim\limits_{x\to+\infty}\frac{x^k}{e^x} = \lim\limits_{x\to+\infty}\frac{k!}{e^x} = 0$

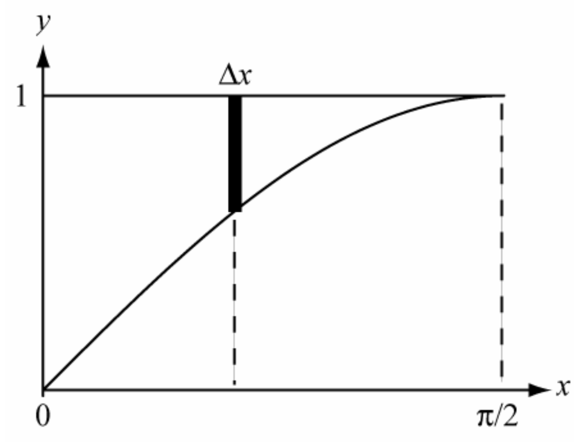
36. E Disks: $\sum \pi(R^2 - r^2)\Delta x$ where $R = 1,\ r = \sin x$

Volume = $\pi\int_0^{\pi/2}(1-\sin^2 x)\,dx$

Note that the expression in (E) can also be written as

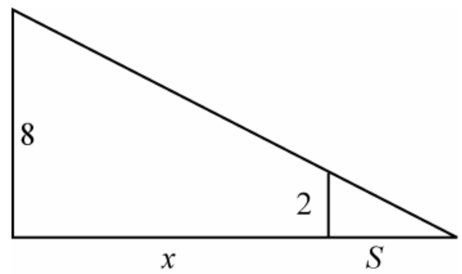
$$\pi\int_0^{\pi/2}\cos^2 x\,dx = -\pi\int_{\pi/2}^{0}\cos^2\left(\tfrac{\pi}{2}-x\right)dx$$
$$= \pi\int_0^{\pi/2}\sin^2 x\,dx$$

and therefore option (D) is also a correct answer.



37. D $\dfrac{x+S}{8} = \dfrac{S}{2} \Rightarrow x = 3S$

$\dfrac{dx}{dt} = 3\dfrac{dS}{dt} = 3\cdot\dfrac{4}{9} = \dfrac{4}{3}$



38. C Check $x=-1,\ \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$ which is convergent by alternating series test

Check $x = 1,\ \sum_{n=1}^{\infty}\dfrac{1}{n}$ which is the harmonic series and known to diverge.

39. C $\dfrac{dy}{y} = \sec^2 x\,dx \Rightarrow \ln|y| = \tan x + k \Rightarrow y = Ce^{\tan x}.\ y(0) = 5 \Rightarrow y = 5e^{\tan x}$

40. E Since f and g are inverses their derivatives at the inverse points are reciprocals. Thus,

$$g'(-2)\cdot f'(5) = 1 \Rightarrow g'(-2) = \frac{1}{-\frac{1}{2}} = -2$$

41. B Take the interval $[0,1]$ and divide it into n pieces of equal length and form the right Riemann Sum for the function $f(x) = \sqrt{x}$. The limit of this sum is what is given and its value is given by $\int_0^1 \sqrt{x}\,dx$

42. A Let $5-x=u$, $dx=-du$, substitute

$$\int_1^4 f(5-x)\,dx=\int_4^1 f(u)(-du)=\int_1^4 f(u)\,du=\int_1^4 f(x)\,dx=6$$

43. A This is an example of exponential growth, $B=B_0\cdot 2^{t/3}$. Find the value of t so $B=3B_0$.

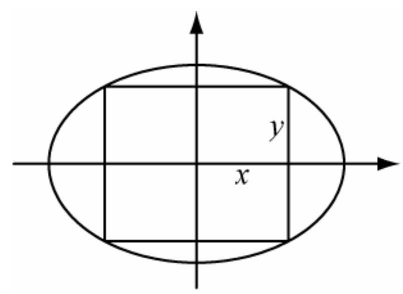
$$3B_0=B_0\cdot 2^{t/3}\Rightarrow 3=2^{t/3}\Rightarrow \ln 3=\frac{t}{3}\ln 2\Rightarrow t=\frac{3\ln 3}{\ln 2}$$

44. A I. Converges by Alternate Series Test

II Diverges by the nth term test: $\lim_{n\to\infty}\frac{1}{n}\left(\frac{3}{2}\right)^n\neq 0$

III Diverges by Integral test: $\int_2^\infty \frac{1}{x\ln x}dx=\lim_{L\to\infty}\ln(\ln x)\Big|_2^L=\infty$

45. B $A=(2x)(2y)=4xy$ and $y=\sqrt{4-\frac{4}{9}x^2}$.



So $A=8x\sqrt{1-\frac{1}{9}x^2}$.

$$A'=8\left(\left(1-\frac{1}{9}x^2\right)^{1/2}+\frac{1}{2}x\left(1-\frac{1}{9}x^2\right)^{-1/2}\left(-\frac{2}{9}x\right)\right)$$

$$=\frac{8}{9}\left(1-\frac{1}{9}x^2\right)^{-1/2}(9-2x^2)$$

$A'=0$ at $x=\pm 3,\frac{3}{\sqrt{2}}$. The maximum area occurs when $x=\frac{3}{\sqrt{2}}$ and $y=\sqrt{2}$. The value of the largest area is $A=4xy=4\cdot\frac{3}{\sqrt{2}}\cdot\sqrt{2}=12$

1. A $\int_0^1 (x - x^2)\,dx = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

2. C $\lim\limits_{x\to 0} \frac{2x^2 + 1 - 1}{x^2} = 2$

3. E $Q'(x) = p(x) \Rightarrow$ degree of Q is $n+1$

4. B If $x = 2$ then $y = 5$. $x\frac{dy}{dt} + y\frac{dx}{dt} = 0$; $2(3) + 5\cdot\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = -\frac{6}{5}$

5. D $r = 2\sec\theta$; $r\cos\theta = 2 \Rightarrow x = 2$. This is a vertical line through the point $(2,0)$.

6. A $\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 3t^2$ thus $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t;\ \frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}$

7. A $\int_0^1 x^3 e^{x^4}\,dx = \frac{1}{4}\int_0^1 e^{x^4}(4x^3 dx) = \frac{1}{4}e^{x^4}\Big|_0^1 = \frac{1}{4}(e-1)$

8. B $f(x) = \ln e^{2x} = 2x,\ f'(x) = 2$

9. D $f'(x) = \frac{2}{3}\cdot\frac{1}{x^{1/3}}$. This does not exist at $x = 0$. D is false, all others are true.

10. E I. $\ln x$ is continuous for $x > 0$
II. e^x is continuous for all x
III. $\ln(e^x - 1)$ is continuous for $x > 0$.

11. E $\int_4^\infty \frac{-2x}{\sqrt[3]{9 - x^2}}\,dx = \lim\limits_{b\to\infty} \frac{3}{2}\left(9 - x^2\right)^{2/3}\Big|_4^b$. This limit diverges. Another way to see this without doing the integration is to observe that the denominator behaves like $x^{2/3}$ which has a smaller degree than the degree of the numerator. This would imply that the integral will diverge.

12. E $v(t) = 2\cos 2t + 3\sin 3t,\ a(t) = -4\sin 2t + 9\cos 3t,\ a(\pi) = -9$.

13. C $\frac{dy}{y} = x^2 dx,\ \ln|y| = \frac{1}{3}x^3 + C_1,\ y = Ce^{\frac{1}{3}x^3}$. Only C is of this form.

14. B The only place that $f'(x)$ changes sign from positive to negative is at $x = -3$.

15. D $f'(x) = e^{\tan^2 x} \cdot \frac{d\left(\tan^2 x\right)}{dx} = 2\tan x \cdot \sec^2 x \cdot e^{\tan^2 x}$

16. A I. Compare with p-series, $p = 2$

II. Geometric series with $r = \frac{6}{7}$

III. Alternating harmonic series

17. A Using implicit differentiation, $\frac{y + xy'}{xy} = 1$. When $x = 1$, $\frac{y + y'}{y} = 1 \Rightarrow y' = 0$.

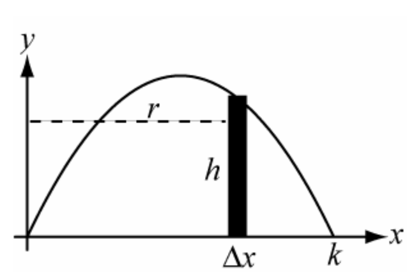
Alternatively, $xy = e^x$, $y = \frac{e^x}{x}$, $y' = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$. $y'(1) = 0$

18. B $f'(x) \cdot e^{f(x)} = 2x \Rightarrow f'(x) = \frac{2x}{e^{f(x)}} = \frac{2x}{1 + x^2}$

19. B Use cylindrical shells which is no longer part of the AP Course Description. Each shell is of the form $2\pi rh\Delta x$ where $r = x$ and $h = kx - x^2$. Solve the equation

$$10 = 2\pi\int_0^k x(kx - x^2)\,dx = 2\pi\left(\frac{kx^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^k = 2\pi \cdot \frac{k^4}{12}.$$

$k = \sqrt[4]{\frac{60}{\pi}} \approx 2.0905$.



20. E $v(t) = -\frac{1}{2}e^{-2t} + 3$ and $x(t) = \frac{1}{4}e^{-2t} + 3t + 4$

21. A Use logarithms.
$\ln y = \frac{1}{3}\ln\left(x^2+8\right) - \frac{1}{4}\ln(2x+1);\ \frac{y'}{y} = \frac{2x}{3\left(x^2+8\right)} - \frac{2}{4(2x+1)};$ at $(0,2),\ y' = -1$.

22. B $f'(x) = x^2e^x + 2xe^x = xe^x(x+2);\ f'(x) < 0$ for $-2 < x < 0$

23. D $L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^4 \sqrt{4t^2+1}\,dt$

24. C This is L'Hôpital's Rule.

25. D At $t = 3$, slope $= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-te^{-t}+e^{-t}}{e^t} = \left.\frac{1-t}{e^{2t}}\right|_{t=3} = -\frac{2}{e^6} = -0.005$

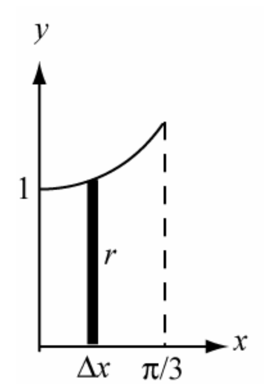
26. B $\frac{2e^{2x}}{1+(e^{2x})^2} = \frac{2e^{2x}}{1+e^{4x}}$

27. C This is a geometric series with $r = \frac{x-1}{3}$. Convergence for $-1 < r < 1$. Thus the series is convergent for $-2 < x < 4$.

28. A $v = \left(\frac{2t+2}{t^2+2t}, 4t\right),\ v(2) = \left(\frac{6}{8}, 8\right) = \left(\frac{3}{4}, 8\right)$

29. E Use the technique of antiderivatives by parts: $u = x$ and $dv = \sec^2 x\,dx$
$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln\left|\cos x\right| + C$

30. C Each slice is a disk with radius $r = \sec x$ and width Δx.
Volume $= \pi\int_0^{\pi/3} \sec^2 x\,dx = \pi\tan x\Big|_0^{\pi/3} = \pi\sqrt{3}$



31. A $s_n = \frac{1}{5}\left(\frac{5+n}{4+n}\right)^{100}$, $\lim_{n\to\infty} s_n = \frac{1}{5}\cdot 1 = \frac{1}{5}$

32. B Only II is true. To see that neither I nor III must be true, let $f(x)=1$ and let $g(x)=x^2-\frac{128}{15}$ on the interval [0, 5].

33. A The value of this integral is 2. Option A is also 2 and none of the others have a value of 2. Visualizing the graphs of $y=\sin x$ and $y=\cos x$ is a useful approach to the problem.

34. E Let $y=PR$ and x=RQ.
$x^2+y^2=40^2,\ 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0,\ x\cdot\frac{3}{4}\left(-\frac{dy}{dt}\right)+y\frac{dy}{dt}=0 \Rightarrow y=\frac{3}{4}x$.
Substitute into $x^2+y^2=40^2$. $x^2+\frac{9}{16}x^2=40^2,\ \frac{25}{16}x^2=40^2,\ x=32$

35. A Apply the Mean Value Theorem to *F*. $F'(c)=\frac{F(b)-F(a)}{b-a}=\frac{0}{a}=0$. This means that there is number in the interval (a,b) for which F' is zero. However, $F'(x)=f(x)$. So, $f(x)=0$ for some number in the interval (a,b).

36. E $v=\pi r^2 h$ and $h+2\pi r=30 \Rightarrow v=2\pi\left(15r^2-\pi r^3\right)$ for $0<r<\frac{15}{\pi}$; $\frac{dv}{dr}=6\pi r\left(10-\pi r\right)$. The maximum volume is when $r=\frac{10}{\pi}$ because $\frac{dv}{dr}>0$ on $\left(0,\frac{10}{\pi}\right)$ and $\frac{dv}{dr}<0$ on $\left(\frac{10}{\pi},\frac{15}{\pi}\right)$.

37. B $\int_0^e f(x)\,dx=\int_0^1 x\,dx+\int_1^e \frac{1}{x}\,dx=\frac{1}{2}+\ln e=\frac{3}{2}$

38. C $\frac{dN}{dt}=kN \Rightarrow N=Ce^{kt}$. $N(0)=1000 \Rightarrow C=1000$. $N(7)=1200 \Rightarrow k=\frac{1}{7}\ln(1.2)$. Therefore $N(12)=1000e^{\frac{12}{7}\ln(1.2)}\approx 1367$.

39. C Want $\frac{y(4)-y(1)}{4-1}$ where $y(x)=\ln|x|+C$. This gives $\frac{\ln 4-\ln 1}{3}=\frac{1}{3}\ln 4=\frac{1}{3}\ln 2^2=\frac{2}{3}\ln 2$.

40. C The interval is $[0,2]$, $x_0=0,\ x_1=1,\ x_2=2$. $S=\frac{1}{3}\left(0+4\ln 2+0\right)=\frac{4}{3}\ln 2$. Note that Simpson's rule is no longer part of the BC Course Description.

41. C $f'(x) = (2x-3)e^{(x^2-3x)^2}$; $f' < 0$ for $x < \frac{3}{2}$ and $f' > 0$ for $x > \frac{3}{2}$.

Thus f has its absolute minimum at $x = \frac{3}{2}$.

42. E Suppose $\lim_{x\to 0} \ln\left((1+2x)^{\csc x}\right) = A$. The answer to the given question is e^A.

Use L'Hôpital's Rule: $\lim_{x\to 0} \ln\left((1+2x)^{\csc x}\right) = \lim_{x\to 0} \frac{\ln(1+2x)}{\sin x} = \lim_{x\to 0} \frac{2}{1+2x} \cdot \frac{1}{\cos x} = 2$.

43. A $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \Rightarrow \sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$

44. E By the Intermediate Value Theorem there is a c satisfying $a < c < b$ such that $f(c)$ is equal to the average value of f on the interval $[a,b]$. But the average value is also given by $\frac{1}{b-a}\int_a^b f(x)\,dx$. Equating the two gives option E.

Alternatively, let $F(t) = \int_a^t f(x)\,dx$. By the Mean Value Theorem, there is a c satisfying $a < c < b$ such that $\frac{F(b)-F(a)}{b-a} = F'(c)$. But $F(b) - F(a) = \int_a^b f(x)\,dx$, and $F'(c) = f(c)$ by the Fundamental Theorem of Calculus. This again gives option E as the answer. This result is called the Mean Value Theorem for Integrals.

45. D This is an infinite geometric series with a first term of $\sin^2 x$ and a ratio of $\sin^2 x$.

The series converges to $\frac{\sin^2 x}{1-\sin^2 x} = \tan^2 x$ for $x \neq (2k+1)\frac{\pi}{2}$, k an integer. The answer is therefore $\tan^2 1 = 2.426$.

1997 Calculus BC Solutions: Part A

1. C $\int_0^1 \sqrt{x}(x+1)\,dx = \int_0^1 x^{\frac{3}{2}} + x^{\frac{1}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{16}{15}$

2. E $x = e^{2t}$, $y = \sin(2t)$; $\frac{dy}{dx} = \frac{2\cos(2t)}{2e^{2t}} = \frac{\cos(2t)}{e^{2t}}$

3. A $f(x) = 3x^5 - 4x^3 - 3x$; $f'(x) = 15x^4 - 12x^2 - 3 = 3(5x^2+1)(x^2-1) = 3(5x^2+1)(x+1)(x-1)$; f' changes from positive to negative only at $x = -1$.

4. C $e^{\ln x^2} = x^2$; so $xe^{\ln x^2} = x^3$ and $\frac{d}{dx}(x^3) = 3x^2$

5. C $f(x) = (x-1)^{\frac{3}{2}} + \frac{1}{2}e^{x-2}$; $f'(x) = \frac{3}{2}(x-1)^{\frac{1}{2}} + \frac{1}{2}e^{x-2}$; $f'(2) = \frac{3}{2} + \frac{1}{2} = 2$

6. A $y = (16-x)^{\frac{1}{2}}$; $y' = -\frac{1}{2}(16-x)^{-\frac{1}{2}}$; $y'(0) = -\frac{1}{8}$; The slope of the normal line is 8.

7. C The slope at $x = 3$ is 2. The equation of the tangent line is $y - 5 = 2(x-3)$.

8. E Points of inflection occur where f' changes from increasing to decreasing, or from decreasing to increasing. There are six such points.

9. A f increases for $0 \le x \le 6$ and decreases for $6 \le x \le 8$. By comparing areas it is clear that f increases more than it decreases, so the absolute minimum must occur at the left endpoint, $x = 0$.

10. B $y = xy + x^2 + 1$; $y' = xy' + y + 2x$; at $x = -1, y = 1$; $y' = -y' + 1 - 2 \Rightarrow y' = -\frac{1}{2}$

11. C $\int_1^\infty x(1+x^2)^{-2}\,dx = \lim_{L\to\infty} -\frac{1}{2}(1+x^2)^{-1}\Big|_1^L = \lim_{L\to\infty} \frac{1}{4} - \frac{1}{2(1+L^2)} = \frac{1}{4}$

12. A f' changes from positive to negative once and from negative to positive twice. Thus one relative maximum and two relative minimums.

13. B $a(t) = 2t - 7$ and $v(0) = 6$; so $v(t) = t^2 - 7t + 6 = (t-1)(t-6)$. Movement is right then left with the particle changing direction at $t = 1$, , therefore it will be farthest to the right at $t = 1$.

14. C Geometric Series. $r = \frac{3}{8} < 1 \Rightarrow$ convergence. $a = \frac{3}{2}$ so the sum will be $S = \frac{\frac{3}{2}}{1 - \frac{3}{8}} = 2.4$

15. D $x = \cos^3 t, y = \sin^3 t$ for $0 \le t \le \frac{\pi}{2}$. $L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$L = \int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt = \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$$

16. B $\lim_{h\to 0} \frac{e^h - 1}{2h} = \frac{1}{2} \lim_{h\to 0} \frac{e^h - e^0}{h} = \frac{1}{2} f'(0)$, where $f(x) = e^x$ and $f'(0) = 1$. $\lim_{h\to 0} \frac{e^h - 1}{2h} = \frac{1}{2}$

17. B $f(x) = \ln(3 - x);\ f'(x) = \frac{1}{x-3},\ f''(x) = -\frac{1}{(x-3)^2},\ f'''(x) = \frac{2}{(x-3)^3};$

$f(2) = 0,\ f'(2) = -1,\ f''(2) = -1,\ f'''(2) = -2;\ \ a_0 = 0, a_1 = -1, a_2 = -\frac{1}{2}, a_3 = -\frac{1}{3}$

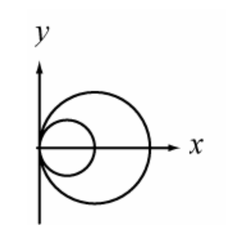
$f(x) \approx -(x-2) - \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3}$

18. C $x = t^3 - t^2 - 1,\ y = t^4 + 2t^2 - 8t;\ \ \frac{dy}{dx} = \frac{4t^3 + 4t - 8}{3t^2 - 2t} = \frac{4t^3 + 4t - 8}{t(3t-2)}$. Vertical tangents at $t = 0, \frac{2}{3}$

19. D $\int_{-4}^{4} f(x)dx - 2\int_{-1}^{4} f(x)dx = (A_1 - A_2) - 2(-A_2) = A_1 + A_2$

20. E $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$. The endpoints of the interval of convergence are when $(x-2) = \pm 3;\ x = -1, 5$.
Check endpoints: $x = -1$ gives the alternating harmonic series which converges. $x = 5$ gives the harmonic series which diverges. Therefore the interval is $-1 \le x < 5$.

21. A Area $= 2 \cdot \frac{1}{2} \int_0^{\pi/2} ((2\cos\theta)^2 - \cos^2\theta)\, d\theta = \int_0^{\pi/2} 3\cos^2\theta\ d\theta$



22. C $g'(x) = f(x)$. The only critical value of g on (a, d) is at $x = c$. Since g' changes from positive to negative at $x = c$, the absolute maximum for g occurs at this relative maximum.

23. E $x = 5\sin\theta;\ \frac{dx}{dt} = 5\cos\theta \cdot \frac{d\theta}{dt};$ When $x = 3, \cos\theta = \frac{4}{5};\ \frac{dx}{dt} = 5\left(\frac{4}{5}\right)(3) = 12$

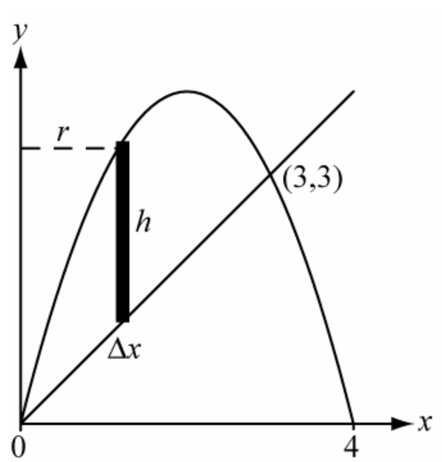
24. D $f'(x) = \sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \cdots = x^2 - \frac{1}{6}x^6 + \cdots \Rightarrow f(x) = \frac{1}{3}x^3 - \frac{1}{42}x^7 + \cdots$ The coefficieint of x^7 is $-\frac{1}{42}$.

25. A This is the limit of a right Riemann sum of the function $f(x) = \sqrt{x}$ on the interval $[a,b]$, so

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{x_i}\,\Delta x = \int_a^b \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_a^b = \frac{2}{3}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$$

76. D Sequence I $\to \frac{5}{2}$; sequence II $\to \infty$; sequence III $\to 1$. Therefore I and III only.

77. E Use shells (which is no longer part of the AP Course Description.)



$\sum 2\pi r h \Delta x$ where $r = x$ and $h = 4x - x^2 - x$

Volume =
$2\pi\int_0^3 x(4x - x^2 - x)\,dx = 2\pi\int_0^3 (3x^2 - x^3)\,dx$

78. A $\lim_{h\to 0} \frac{\ln(e+h)-1}{h} = \lim_{h\to 0} \frac{\ln(e+h)-\ln e}{h} = f'(e)$ where $f(x) = \ln x$

79. D Count the number of places where the graph of $y(t)$ has a horizontal tangent line. Six places.

80 B Find the first turning point on the graph of $y = f'(x)$. Occurs at $x = 0.93$.

81. D f assumes every value between -1 and 3 on the interval $(-3, 6)$. Thus $f(c) = 1$ at least once.

82. B $\int_0^x (t^2 - 2t)\,dt \ge \int_2^x t\,dt$; $\frac{1}{3}x^3 - x^2 \ge \frac{1}{2}x^2 - 2$. Using the calculator, the greatest x value on the interval $[0, 4]$ that satisfies this inequality is found to occur at $x = 1.3887$.

83. E $\frac{dy}{y} = (1 + \ln x)\,dx$; $\ln|y| = x + x\ln x - x + k = x\ln x + k$; $|y| = e^k e^{x\ln x} \Rightarrow y = Ce^{x\ln x}$. Since $y = 1$ when $x = 1$, $C = 1$. Hence $y = e^{x\ln x}$.

84. C $\int x^2 \sin x \, dx$; Use the technique of antiderivatives by parts with $u = x^2$ and $dv = \sin x \, dx$. It will take 2 iterations with a different choice of u and dv for the second iteration.

$$\begin{aligned}\int x^2 \sin x \, dx &= -x^2 \cos x + \int 2x \cos x \, dx \\ &= -x^2 \cos x + \left(2x \sin x - \int 2 \sin x \, dx\right) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C\end{aligned}$$

85. D I. Average rate of change of f is $\frac{f(3) - f(1)}{3 - 1} = \frac{5}{2}$. True

II. Not enough information to determine the average value of f. False

III. Average value of f' is the average rate of change of f. True

86. A Use partial fractions. $\frac{1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$; $1 = A(x+3) + B(x-1)$

Choose $x = 1 \Rightarrow A = \frac{1}{4}$ and choose $x = -3 \Rightarrow B = -\frac{1}{4}$.

$$\int \frac{1}{(x-1)(x+3)} dx = \frac{1}{4}\left[\int \frac{1}{x-1} dx - \int \frac{1}{x+3} dx\right] = \frac{1}{4} \ln\left|\frac{x-1}{x+3}\right| + C$$

87. B Squares with sides of length x. Volume = $\int_0^2 x^2 dy = \int_0^2 (2 - y)\, dy$

88. C $f(x) = \int_0^{x^2} \sin t \, dt$; $f'(x) = 2x \sin(x^2)$; For the average rate of change of f we need to determine $f(0)$ and $f(\sqrt{\pi})$. $f(0) = 0$ and $f\left(\sqrt{\pi}\right) = \int_0^{\pi} \sin t \, dt = 2$. The average rate of change of f on the interval is $\frac{2}{\sqrt{\pi}}$. See how many points of intersection there are for the graphs of $y = 2x \sin(x^2)$ and $y = \frac{2}{\sqrt{\pi}}$ on the interval $\left[0, \sqrt{\pi}\right]$. There are two.

89. D $f(x)=\int_1^x \frac{t^2}{1+t^5}\,dt;\ f(4)=\int_1^4 \frac{t^2}{1+t^5}\,dt=0.376$

Or, $f(4)=f(1)+\int_1^4 \frac{x^2}{1+x^5}\,dx=0.376$

Both statements follow from the Fundamental Theorem of Calculus.

90. B $F(x)=kx;\ 10=4k \Rightarrow k=\frac{5}{2};$ Work $=\int_0^6 F(x)dx=\int_0^6 \frac{5}{2}x\,dx=\frac{5}{4}x^2\Big|_0^6=45$ inch-lbs

1. C f will be increasing when its derivative is positive.
$f'(x)=3x^2+6x-9=3(x^2+2x-3)$ $f'(x)=3(x+3)(x-1)>0$ for $x<-3$ or $x>1$.

2. A $\frac{dx}{dt}=5$ and $\frac{dy}{dt}=3 \Rightarrow \frac{dy}{dx}=\frac{3}{5}$

3. D Find the derivative implicitly and substitute. $2y\cdot y'+3(xy+1)^2(x\cdot y'+y)=0$;
$2(-1)\cdot y'+3((2)(-1)+1)^2((2)\cdot y'+(-1))=0$; $-2y'+6\cdot y'-3=0$; $y'=\frac{3}{4}$

4. A Use partial fractions. $\frac{1}{x^2-6x+8}=\frac{1}{(x-4)(x-2)}=\frac{1}{2}\left(\frac{1}{x-4}-\frac{1}{x-2}\right)$

$$\int \frac{1}{x^2-6x+8}\,dx=\frac{1}{2}\left(\ln|x-4|-\ln|x-2|\right)+C=\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right|+C$$

5. A $h'(x)=f'\left(g(x)\right)\cdot g'(x)$; $h''(x)=f''\left(g(x)\right)\cdot g'(x)\cdot g'(x)+f'\left(g(x)\right)\cdot g''(x)$
$h''(x)=f''\left(g(x)\right)\cdot\left(g'(x)\right)^2+f'\left(g(x)\right)\cdot g''(x)$

6. E The graph of *h* has 2 turning points and one point of inflection. The graph of h' will have 2 *x*-intercepts and one turning point. Only (C) and (E) are possible answers. Since the first turning point on the graph of *h* is a relative maximum, the first zero of h' must be a place where the sign changes from positive to negative. This is option (E).

7. E $\int_1^e \frac{x^2-1}{x}\,dx=\int_1^e x-\frac{1}{x}\,dx=\left(\frac{1}{2}x^2-\ln x\right)\Bigg|_1^e=\left(\frac{1}{2}e^2-1\right)-\left(\frac{1}{2}-0\right)=\frac{1}{2}e^2-\frac{3}{2}$

8. B $y(x)=-\frac{1}{3}(\cos x)^3+C$; Let $x=\frac{\pi}{2}$, $0=-\frac{1}{3}\left(\cos\frac{\pi}{2}\right)^3+C\Rightarrow C=0$. $y(0)=-\frac{1}{3}(\cos 0)^3=-\frac{1}{3}$

9. D Let $r(t)$ be the rate of oil flow as given by the graph, where *t* is measured in hours. The total number of barrels is given by $\int_0^{24} r(t)dt$. This can be approximated by counting the squares below the curve and above the horizontal axis. There are approximately five squares with area 600 barrels. Thus the total is about $3,000$ barrels.

10. E $v(t)=\left(3t^2-1, 6(2t-1)^2\right)$ and $a(t)=\left(6t, 24(2t-1)\right)\Rightarrow a(1)=(6,24)$

11. A Since f is linear, its second derivative is zero and the integral gives the area of a rectangle with zero height and width $(b-a)$. This area is zero.

12. E $\lim_{x\to 2^-} f(x) = \ln 2 \neq 4\ln 2 = \lim_{x\to 2^+} f(x)$. Therefore the limit does not exist.

13. B At $x = 0$ and $x = 2$ only. The graph has a non-vertical tangent line at every other point in the interval and so has a derivative at each of these other x's.

14. E $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$; $\sin 1 \approx 1 - \frac{1^3}{3!} + \frac{1^5}{5!} = 1 - \frac{1}{6} + \frac{1}{120}$

15. B Use the technique of antiderivatives by parts. Let $u = x$ and $dv = \cos x\, dx$.

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

16. C Inflection point will occur when f'' changes sign. $f'(x) = 15x^4 - 20x^3$.
$f''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$. The only sign change is at $x = 1$.

17. D From the graph $f(1) = 0$. Since $f'(1)$ represents the slope of the graph at $x = 1$, $f'(1) > 0$. Also, since $f''(1)$ represents the concavity of the graph at $x = 1$, $f''(1) < 0$.

18. B
I. Divergent. The limit of the nth term is not zero.
II. Convergent. This is the same as the alternating harmonic series.
III. Divergent. This is the harmonic series.

19. D Find the points of intersection of the two curves to determine the limits of integration.
$4\sin\theta = 2$ when $\sin\theta = 0.5$; this is at $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Area $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left((4\sin\theta)^2 - (2)^2 \right) d\theta$

20. E $\left.\frac{d\left(\sqrt[3]{x}\right)}{dt}\right|_{x=8} = \left.\frac{1}{3}x^{-\frac{2}{3}} \cdot \frac{dx}{dt}\right|_{x=8} = \frac{1}{3}(8)^{-\frac{2}{3}} \cdot \frac{dx}{dt} = \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{dx}{dt} = \frac{1}{12}\cdot\frac{dx}{dt} \Rightarrow k = 12$

21 C The length of this parametric curve is given by $\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{(t^2)^2 + t^2}\,dt$.

22. A This is the integral test applied to the series in (A). Thus the series in (A) converges. None of the others must be true.

23. E I. False. The relative maximum could be at a cusp.
II. True. There is a critical point at $x = c$ where $f'(c)$ exists
III. True. If $f''(c) > 0$, then there would be a relative minimum, not maximum

24. C All slopes along the diagonal $y = -x$ appear to be 0. This is consistent only with option (C). For each of the others you can see why they do not work. Option (A) does not work because all slopes at points with the same x coordinate would have to be equal. Option (B) does not work because all slopes would have to be positive. Option (D) does not work because all slopes in the third quadrant would have to be positive. Option (E) does not work because there would only be slopes for $y > 0$.

25. C $\int_0^\infty x^2 e^{-x^3}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x^3}\,dx = \lim_{b\to\infty} -\frac{1}{3}e^{-x^3}\Big|_0^b = \frac{1}{3}$.

26. E As $\lim_{t\to\infty}\frac{dP}{dt} = 0$ for a population satisfying a logistic differential equation, this means that $P \to 10{,}000$.

27. D If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

$$f'(1) = \sum_{n=1}^{\infty} n a_n 1^{n-1} = \sum_{n=1}^{\infty} n a_n$$

28. C Apply L'Hôpital's rule. $\lim_{x\to 1}\frac{\int_1^x e^{t^2}\,dt}{x^2 - 1} = \lim_{x\to 1}\frac{e^{x^2}}{2x} = \frac{e}{2}$

76. D The first series is either the harmonic series or the alternating harmonic series depending on whether k is odd or even. It will converge if k is odd. The second series is geometric and will converge if $k < 4$.

77. E $f'(t) = \left(-e^{-t}, -\sin t\right)$; $f''(t) = \left(e^{-t}, -\cos t\right)$.

78. B $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. However, $C = 2\pi r$ and $\frac{dr}{dt} = -0.1$. Thus $\frac{dA}{dt} = -0.1C$.

79. A None. For every positive value of a the denominator will be zero for some value of x.

80. B The area is given by $\int_{-\frac{2}{3}}^{\frac{2}{3}} \left(1 + \ln(\cos^4 x)\right) dx = 0.919$

81. B $\frac{dy}{dx} = \sqrt{1 - y^2}$; $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left((1 - y^2)^{\frac{1}{2}}\right) = \frac{1}{2}\left(1 - y^2\right)^{-\frac{1}{2}} \cdot (-2y) \cdot \frac{dy}{dx} = -y$

82. B $\int_3^5 \left[f(x) + g(x)\right] dx = \int_3^5 \left[2g(x) + 7\right] dx = 2\int_3^5 g(x)\,dx + (7)(2) = 2\int_3^5 g(x)\,dx + 14$

83. C Use a calculator. The maximum for $\left|\ln x - \left(\frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}\right)\right|$ on the interval $0.3 \le x \le 1.7$ occurs at $x = 0.3$.

84. B You may use the ratio test. However, the series will converge if the numerator is $(-1)^n$ and diverge if the numerator is 1^n. Any value of x for which $|x + 2| > 1$ in the numerator will make the series diverge. Hence the interval is $-3 \le x < -1$.

85. C There are 3 trapezoids. $\frac{1}{2} \cdot 3\left(f(2) + f(5)\right) + \frac{1}{2} \cdot 2\left(f(5) + f(7)\right) + \frac{1}{2} \cdot 1\left(f(7) + f(8)\right)$

86. C Each cross section is a semicircle with a diameter of y. The volume would be given by

$$\int_0^8 \frac{1}{2}\pi\left(\frac{y}{2}\right)^2 dx = \frac{\pi}{8}\int_0^8 \left(\frac{8-x}{2}\right)^2 dx = 16.755$$

87. D Find the x for which $f'(x)=1$. $f'(x)=4x^3+4x=1$ only for $x=0.237$. Then $f(0.237)=0.115$. So the equation is $y-0.115=x-0.237$. This is equivalent to option (D).

88. C From the given information, f is the derivative of g. We want a graph for f that represents the slopes of the graph g. The slope of g is zero at a and b. Also the slope of g changes from positive to negative at one point between a and b. This is true only for figure (C).

89. A The series is the Maclaurin expansion of e^{-x}. Use the calculator to solve $e^{-x}=x^3$.

90. A Constant acceleration means linear velocity which in turn leads to quadratic position. Only the graph in (A) is quadratic with initial $s=2$.

91. E $v(t)=11+\int_0^t a(x)\,dx \approx 11+\left[2\cdot 5+2\cdot 2+2\cdot 8\right]=41$ ft/sec.

92. D $f'(x)=2x-2,\ f'(2)=2,$ and $f(2)=3$, so an equation for the tangent line is $y=2x-1$. The difference between the function and the tangent line is represented by $(x-2)^2$. Solve $(x-2)^2<0.5$. This inequality is satisfied for all x such that $2-\sqrt{0.5}<x<2+\sqrt{0.5}$. This is the same as $1.293<x<2.707$. Thus the largest value in the list that satisfies the inequality is 2.7.