Sample Examination II

Section I Part A

Directions: Solve each of the following problems, using available space for scratchwork. After examining the form of the choices, decide which is the best of the choices given. Do not spend too much time on any one problem. Calculators may NOT be used on this part of the exam.

In this test: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. $\int_0^2 (2x^3 + 3)\, dx =$

(A) 8

(B) 11

(C) 14

(D) 20

(E) 24

Answer

2. The slope field for the differential equation $\frac{dy}{dx} = \frac{3y}{xy + 5x}$ will have vertical segments when

(A) $x = 0$, only

(B) $y = 0$, only

(C) $y = -5$, only

(D) $y = 5$, only

(E) $x = 0$ or $y = -5$

Answer

3. Suppose that f is a continuous function defined for all real numbers x and $f(-5) = 3$ and $f(-1) = -2$. If $f(x) = 0$ for one and only one value of x, then which of the following could be x?

(A) -7

(B) -2

(C) 0

(D) 1

(E) 2

Answer

4. If $f(x) = (2 + 3x)^4$, then the fourth derivative of f is

(A) 0

(B) $4!(3)$

(C) $4!(3^4)$

(D) $4!(3^5)$

(E) $4!(2 + 3x)$

Answer

5. At what value(s) of x does $f(x) = x^4 - 8x^2$ have a relative minimum?

(A) 0 and -2 only

(B) 0 and 2 only

(C) 0 only

(D) -2 and 2 only

(E) -2, 0, and 2

Answer

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6. $\int \sqrt{x}(x+2)\ dx =$

(A) $\sqrt{x^3} + 2\sqrt{x} + C$ (B) $\frac{2}{5}x^{\frac{3}{2}} + \frac{4}{3}x^{\frac{1}{2}} + C$ (C) $\frac{3}{2}\sqrt{x} + \frac{1}{\sqrt{x}} + C$

(D) $\frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + C$ (E) $\frac{x^2}{2}\left(\frac{2}{3}x^{\frac{3}{2}} + 2x\right) + C$

Answer

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7. The $\lim\limits_{h \to 0} \frac{|x+h| - |x|}{h}$ at $x = 3$ is

(A) -1

(B) 0

(C) 1

(D) 3

(E) nonexistent

Answer

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8. The function $y = x^4 + bx^2 + 8x + 1$ has a horizontal tangent and a point of inflection for the same value of x. What must be the value of b?

(A) -6

(B) -1

(C) 1

(D) 4

(E) 6

Answer

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9. If $y = 7$ is a horizontal asymptote of a rational function f, then which of the following must be true?

(A) $\lim\limits_{x \to 7} f(x) = \infty$

(B) $\lim\limits_{x \to -\infty} f(x) = -7$

(C) $\lim\limits_{x \to 0} f(x) = 7$

(D) $\lim\limits_{x \to 7} f(x) = 0$

(E) $\lim\limits_{x \to \infty} f(x) = 7$

Answer

10. Let $f(x) = e^{(x^2)}$. At how many points in the interval $[-a, a]$, does the instantaneous rate of change of f equal the average rate of change of f?

(A) None

(B) One

(C) Two

(D) Three

(E) More than three

Answer

11. Let f be the function given by $f(x) = x^3$. What are all values of c that satisfy the conclusion of the Mean Value Theorem on the closed interval $[-1, 2]$?

(A) 0 only

(B) 1 only

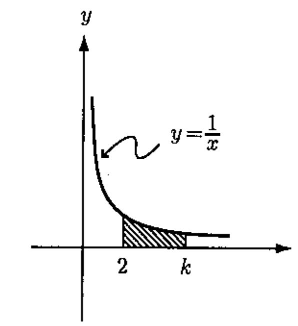
(C) $\sqrt{3}$ only

(D) -1 and 1

(E) $-\sqrt{3}$ and $\sqrt{3}$

Answer

12.



For the figure above, the area of the shaded region is $\ln 4$ when k is

(A) 4

(B) 8

(C) e

(D) e^2

(E) e^3

Answer

13. If $x + y = xy$, then $\frac{dy}{dx}$ is

(A) $\frac{1}{x-1}$

(B) $\frac{y-1}{x-1}$

(C) $\frac{1-y}{x-1}$

(D) $x + y - 1$

(E) $\frac{2-xy}{y}$

Answer

14. If f and g are continuously differentiable functions defined all real numbers, which of the following definite integrals is equal to $f(g(4)) - f(g(2))$?

(A) $\int_2^4 f'(g(x))\,dx$

(B) $\int_2^4 f(g(x))f'(x)\,dx$

(C) $\int_2^4 f(g(x))g'(x)\,dx$

(D) $\int_2^4 f'(g(x))g'(x)\,dx$

(E) $\int_2^4 f(g'(x))g'(x)\,dx$

Answer

15. The velocity of a particle moving along the y-axis is given by $v(t) = 8 - 2t$ for $t \geq 0$. The particle moves upward until it reaches the origin and then moves downward. The position of the particle at any time t is given by

(A) $-t^2 + 8t - 16$

(B) $-t^2 + 8t + 16$

(C) $2t^2 - 8t - 16$

(D) $8t - 2t^2$

(E) $8t - t^2$

Answer

16. If the substituion $u = \sqrt{x-1}$ is made, the integral $\int_2^5 \frac{\sqrt{x-1}}{x}\,dx =$

(A) $\int_2^5 \frac{2u^2}{u^2+1}\,du$

(B) $\int_1^2 \frac{u^2}{u^2+1}\,du$

(C) $\int_1^2 \frac{u^2}{2(u^2+1)}\,du$

(D) $\int_2^5 \frac{u}{u^2+1}\,du$

(E) $\int_1^2 \frac{2u^2}{u^2+1}\,du$

Answer

17. What are all values of x for which the function $f(x) = x^3 + 6x^2 + 9x + 1$ is increasing?

(A) $(-\infty, -3)$ only

(B) $(-3, -1)$ only

(C) $(-1, \infty)$ only

(D) $(-\infty, -3) \cup (-1, \infty)$

(E) $(-\infty, -3) \cup (1, \infty)$

Answer

18. If $\int_0^2 (2x^3 - kx^2 + 2k)\, dx = 12$, then k must be

(A) -3

(B) -2

(C) 1

(D) 2

(E) 3

Answer

19. $\int (\sec^2 x)(\tan^2 x)\, dx =$

(A) $\frac{1}{3}\tan^3 x + C$

(B) $\tan^3 x + C$

(C) $\frac{1}{2}\tan^2 x + C$

(D) $\frac{1}{3}\sec^3 x + C$

(E) $\tan^2 x + C$

Answer

20. For $|x| < 1$, the derivative of $y = \ln\sqrt{1-x^2}$ is

(A) $\frac{x}{1-x^2}$ (B) $\frac{x}{x^2-1}$ (C) $\frac{-1}{x^2-1}$ (D) $\frac{1}{2(1-x^2)}$ (E) $\frac{1}{\sqrt{1-x^2}}$

Answer

21. A function whose derivative is a constant multiple of itself must be

(A) periodic

(B) linear

(C) exponential

(D) quadratic

(E) logarithmic

Answer

22. What are all values of x for which the graph of $y = x^3 - 6x^2$ is concave downward?

(A) $0 < x < 4$

(B) $x > 2$

(C) $x < 2$

(D) $x < 0$

(E) $x > 4$

Answer

23. The edge of a cube is increasing at the rate of 0.05 centimeters per second. In terms of the edge of the cube, s, what is the rate of change of the volume of the cube, in cubic centimeters per second?

(A) 0.05^3

(B) $0.05s^2$

(C) $0.05s^3$

(D) $0.15s^2$

(E) $3s^2$

Answer

24. The tangent line to the graph $y = e^{2-x}$ at the point $(1, e)$ intersects both coordinate axes. What is the area of the triangle formed by this tangent line and the coordinate axes?

(A) $2e$

(B) $e^2 - 1$

(C) e^2

(D) $2e\sqrt{e}$

(E) $4e$

Answer

25. If f is the function defined by $f(x) = \frac{5x^7}{7} + 4x^6 + 6x^5 + x + 1$, what are all the x-coordinates of the points of inflection of the graph of f?

(A) -2 only

(B) 0 only

(C) 2 only

(D) -2 and 0 only

(E) -2, 0, and 2

Answer

26. A normal line to the graph of a function f at the point $(x, f(x))$ is defined to be the line perpendicular to the tangent line at that point. The equation of the normal line to the curve $y = \sqrt[3]{x^2 - 1}$ at the point where $x = 3$ is

(A) $y + 12x = 38$

(B) $y - 4x = 10$

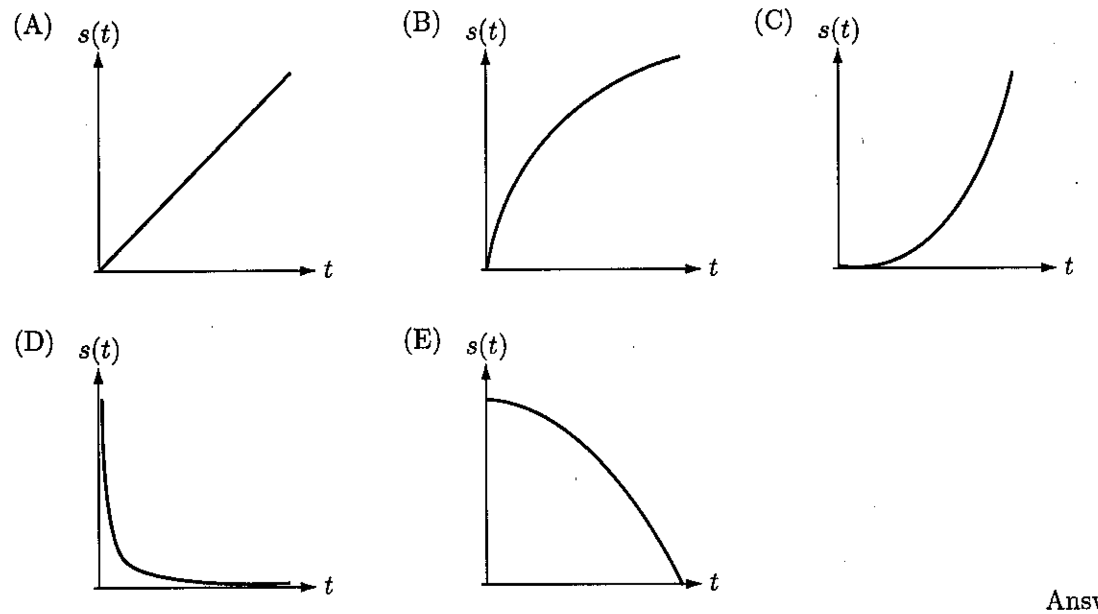
(C) $y + 2x = 4$

(D) $y + 2x = 8$

(E) $y - 2x = -4$

Answer

27. Which graph best represents the position of a particle, $s(t)$, as a function of time, if the particle's velocity and acceleration are both positive?



Answer

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28. If n is a positive integer, then $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right] =$

(A) $\int_0^1 \frac{1}{x^2}\,dx$

(B) $\int_0^1 x^2\,dx$

(C) $\int_0^1 \frac{2}{x^2}\,dx$

(D) $\int_0^1 \frac{1}{x}\,dx$

(E) $\int_0^2 x^2\,dx$

Answer

Section I Part B

Directions: Solve each of the following problems, using available space for scratchwork. After examining the form of the choices, decide which is the best of the choices given. Do not spend too much time on any one problem. Calculators may be used on this part of the exam.

In this test:

(1) The exact numerical value of the correct answer does not always appear among the choices given. When this happens, select from among the choices, the number that best approximates the exact numerical value.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

29. If f is a function such that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0$, which of the following must be true?

(A) $\lim_{x\to a} f(x)$ does not exist

(B) $f(a)$ does not exist

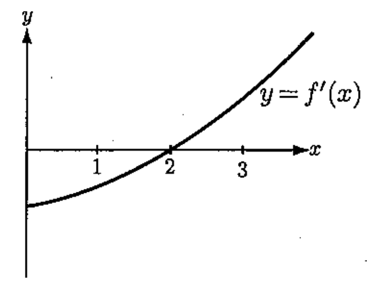
(C) $f'(a) = 0$

(D) $f(a) = 0$

(E) $f(x)$ is continuous at $x = 0$

Answer

30.



The graph of the derivative of a twice-differentiable function f is shown above. If $f(1) = -2$, which of the following is true?

(A) $f(2) < f'(2) < f''(2)$

(B) $f''(2) < f'(2) < f(2)$

(C) $f'(2) < f(2) < f''(2)$

(D) $f(2) < f''(2) < f'(2)$

(E) $f'(2) < f''(2) < f(2)$

Answer

31. Let f be a function that is everywhere differentiable. The value of $f'(x)$ is given for several values of x in the table below.

x	−10	−5	0	5	10
$f'(x)$	−2	−1	0	1	2

If $f'(x)$ is always increasing, which statement about $f(x)$ must be true?

(A) $f(x)$ has a relative minimum at $x = 0$.

(B) $f(x)$ is concave downwards for all x.

(C) $f(x)$ has a point of inflection at $(0, f(0))$.

(D) $f(x)$ passes through the origin.

(E) $f(x)$ is an odd function.

Answer

32. A certain species of fish will grow from x million to $x(15 - x)$ million each year. In order to sustain a steady catch each year, a limit of $x(15 - x) - x$ million fish are to be caught, leaving x million fish to reproduce each year. What is the number of fish which should be left to reproduce each year so that the maximum catch may be sustained from year to year?

(A) 5 million

(B) 7 million

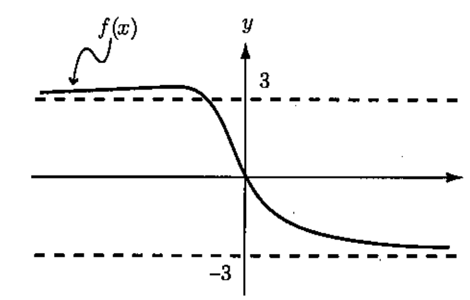
(C) 7.5 million

(D) 10 million

(E) 15 million

Answer

33.



The figure above shows the graph of a function $f(x)$ which has horizontal asymptotes of $y = 3$ and $y = -3$. Which of the following statements are true?

I. $f'(x) < 0$ for all $x \geq 0$

II. $\lim_{x \to +\infty} f'(x) = 0$

III. $\lim_{x \to -\infty} f'(x) = 2$

(A) I only

(B) II only

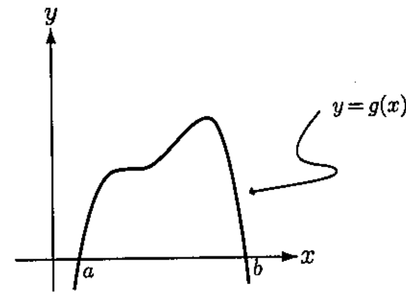
(C) III only

(D) I and II only

(E) I, II, and III

Answer

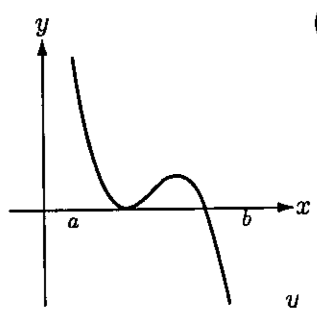
34.



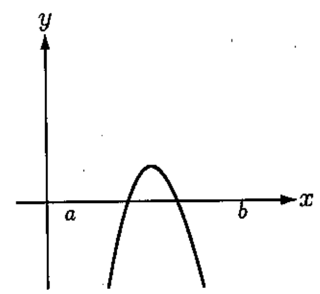
Let $g(x) = \int_a^x f(t)\,dt$, where $a \leq x \leq b$. The figure above shows the graph of g on $[a, b]$.

Which of the following could be the graph of $y = \frac{d}{dx} f(x)$ on $[a, b]$?

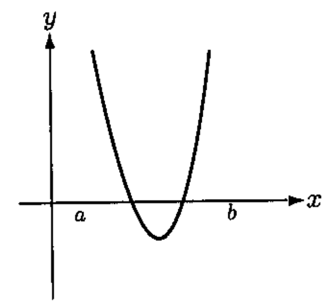
(A)



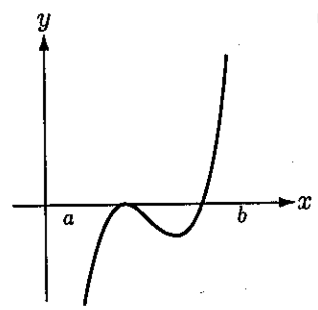
(B)



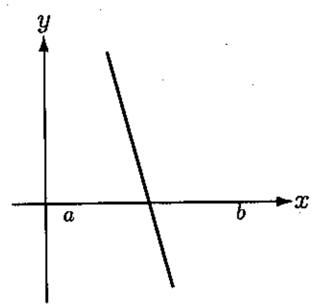
(C)



(D)

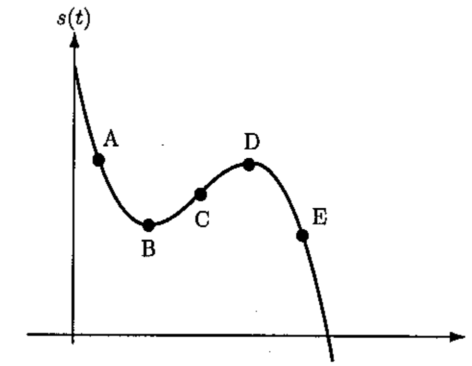


(E)



Answer

35.



The graph above shows the distance $s(t)$ from a reference point of a particle moving on a number line, as a function of time. Which of the points marked is closest to the point where the acceleration first becomes negative?

(A) A

(B) B

(C) C

(D) D

(E) E

Answer

36. The derivative of f is given by $f'(x) = e^x(-x^3 + 3x) - 3$ for $0 \leq x \leq 5$.

At what value of x is $f(x)$ an absolute minimum?

(A) For no value of x

(B) 0

(C) 0.618

(D) 1.623

(E) 5

Answer

37.

x	$f(x)$
3.99800	1.15315
3.99900	1.15548
4.00000	1.15782
4.00100	1.16016
4.00200	1.16250

The table above gives values of a differentiable function f. What is the approximate value of $f'(4)$?

(A) 0.00234

(B) 0.289

(C) 0.427

(D) 2.340

(E) $f'(4)$ cannot be determined from the information given.

Answer

38. Consider the function $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$

In order for $f(x)$ to be continuous at $x = 0$, the value of k must be

(A) 0

(B) 1

(C) −1

(D) π

(E) a number greater than 1

Answer

39. In ther interval $0 \leq x \leq 5$ the graphs of $y = \cos 2x$ and $y = \sin 3x$ intersect four times. Let A, B, C, and D be the x-coordinates of these points so that $0 < A < B < C < D < 5$. Which of the definite integrals below represents the largest number?

(A) $\int_0^A (\cos 2x - \sin 3x)\ dx$

(B) $\int_A^B (\sin 3x - \cos 2x)\ dx$

(C) $\int_B^C (\sin 3x - \cos 2x)\ dx$

(D) $\int_C^D (\cos 2x - \sin 3x)\ dx$

(E) $\int_C^D (\sin 3x - \cos 2x)\ dx$

Answer

40. The function $f(x) = \tan(3^x)$ has one zero in the interval $[0, 1.4]$. The derivative at this point is

(A) 0.411

(B) 1.042

(C) 3.451

(D) 3.763

(E) undefined

Answer

41.

x	0	1	2	3	4	5	6
$f(x)$	0	0.25	0.48	0.68	0.84	0.95	1

For the function whose values are given in the table above, $\int_0^6 f(x)\ dx$ is approximated by a Riemann Sum using the value at the midpoint of each of three intervals of width 2. The approximation is

(A) 2.64

(B) 3.64

(C) 3.72

(D) 3.76

(E) 4.64

Answer

42. $\frac{d}{dx}\int_x^{x^3} \sin(t^2)\ dt =$

(A) $\sin(x^6) - \sin(x^2)$

(B) $6x^2 \sin(x^3) - 2\sin x$

(C) $3x^2 \sin(x^6) - \sin(x^2)$

(D) $6x^5 \sin(x^6) - 2\sin(x^2)$

(E) $2x^3 \cos(x^6) - 2x\cos(x^2)$

Answer

43. A tank is being filled with water at the rate of $300\sqrt{t}$ gallons per hour with $t > 0$ measured in hours. If the tank is originally empty, how many gallons of water are in the tank after 4 hours?

(A) 600

(B) 900

(C) 1200

(D) 1600

(E) 2400

Answer

44. The region in the first quadrant enclosed by the graphs $y = x$ and $y = 2\sin x$ is revolved about the x-axis. The volume of the solid generated is

(A) 1.895

(B) 2.126

(C) 5.811

(D) 6.678

(E) 13.355

Answer

45. If $y = xe^x$, then $\frac{d^n y}{dx^n} =$

(A) e^x

(B) e^{nx}

(C) $(x+n)e^x$

(D) $x^n e^x$

(E) $(x+n^2)e^x$

Answer

SECTION II - FREE-RESPONSE QUESTIONS
GENERAL INSTRUCTIONS

For each part of Section II, you may wish to look over the problems before starting to work on them. It is not expected that everyone will be able to complete all parts of all problems. All problems are given equal weight, but the parts of a particular problem are not necessarily given equal weight.

- YOU SHOULD WRITE ALL WORK FOR EACH PART OF EACH PROBLEM IN THE SPACE PROVIDED FOR THAT PART IN THE PINK TEST BOOKLET. Be sure to write clearly and legibly. If you make an error, you may save time by crossing it out rather than trying to erase it. Erased or crossed-out work will not be graded.
- Show all your work. Clearly label any functions, graphs, tables, or other objects that you use. You will be graded on the correctness and completeness of your methods as well as your answers. Answers without supporting work may not receive credit.
- Justifications require that you give mathematical (noncalculator) reasons.
- Your work must be expressed in standard mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as fnInt(X^2, X, 1, 5).
- Unless otherwise specified, answers (numerical or algebraic) need not be simplified.
- If you use decimal approximations in calculations, you will be graded on accuracy. Unless otherwise specified, your final answers should be accurate to three places after the decimal point.
- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

SECTION II PART A: QUESTIONS 1,2,3

A GRAPHING CALCULATOR IS REQUIRED FOR SOME PROBLEMS OR PARTS OF PROBLEMS IN THIS SECTION OF THE EXAM.

You are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your problem, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results.

YOU HAVE 45 MINUTES FOR PART A.

DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

SECTION II PART B: QUESTION 4,5,6

YOU MAY **NOT** USE A GRAPHING CALCULATOR IN THIS PART OF THE EXAM.

YOU HAVE 45 MINUTES FOR PART B.

YOU MAY USE SOME OF THIS TIME TO WORK ON PART A. IF YOU WORK ON PART A, YOU MAY **NOT** USE A GRAPHING CALCULATOR.

Section II Part A: Graphing Calculator MAY BE USED.

1. Let $f(x) = a(36 - x^2)$ for $0 < a < 1$ and let R be the region in the first quadrant bounded by the y-axis, the graph of f, and the graph of $g(x) = 36 - x^2$.

   (a) Find the area of R in terms of a.

   (b) Find, in terms of a, the equation of the line L, tangent to the graph of f at the point $(6, 0)$.

   (c) Determine the value of a such that the line L divides the region R into two parts with equal areas. Show the analysis that leads to your conclusion.

(a) Find the area of R in terms of a.

(b) Find, in terms of a, the equation of the line L, tangent to the graph of f at the point $(6, 0)$.

(c) Determine the value of a such that the line L divides the region R into two parts with equal areas. Show the analysis that leads to your conclusion.

Section II Part A: Graphing Calculator MAY BE USED.

2. The temperature $T(x)$, in °F, in a small office building without air conditioning is given by $T(x) = 73 - 14\cos\left(\frac{\pi(x-3.4)}{12}\right)$ where x is the time elapsed since midnight, $0 \le x \le 24$.

To cool the building, the air conditioning is turned on when the temperature first reaches the desired temperature T_0 and left on until the office closes at 6:00 p.m. (that is, when $x = 18$).

The cost per day, in dollars, of cooling is given by $C(x) = 0.16\int_x^{18}(T(x) - T_0)\,dx$ for $T(x) \ge T_0$.

(a) Estimated to the nearest half-hour, at what time will the temperature first reach 70°F?

(b) Estimated to the nearest half-hour, at what time will the temperature first reach 77°F?

(c) What is the cost per day of cooling the office if the desired temperature is 70°F? Show your reasoning.

(d) How much is saved per day if the desired temperature is raised to 77°F?

(a) Estimated to the nearest half-hour, at what time will the temperature first reach 70°F?

(b) Estimated to the nearest half-hour, at what time will the temperature first reach 77°F?

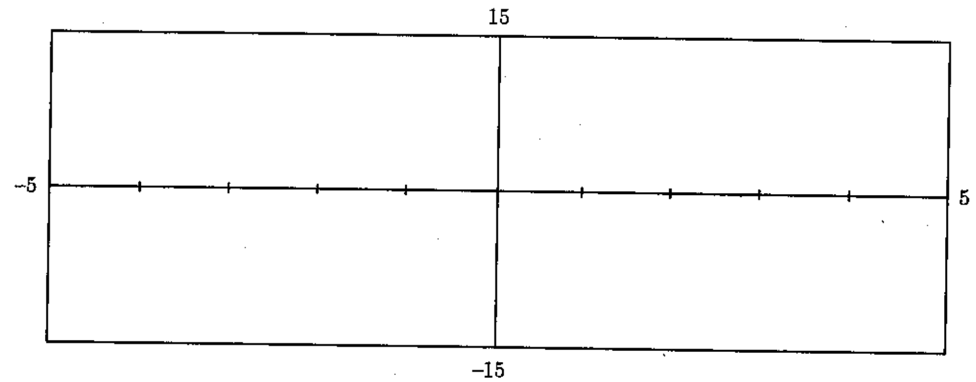
(c) What is the cost per day of cooling the office if the desired temperature is 70°F? Show your reasoning.

(d) How much is saved per day if the desired temperature is raised to 77°F?

Section II Part A: Graphing Calculator MAY BE USED.

3. This problem deals with functions defined by $f(x) = x^3 - 3bx$ with $b > 0$.

   (a) In the viewing window provided below, graph the members of the family $f(x) = x^3 - 3bx$ with $b = 1$, $b = 2$, and $b = 3$. Label each graph.



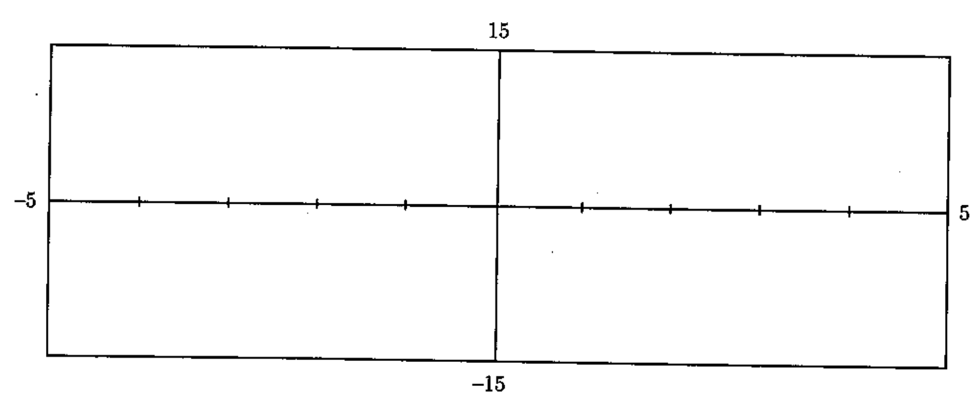
Viewing Window
$[-5, 5] \times [-15, 15]$

   (b) Find the x- and y-coordinates of the relative maximum points of f in terms of b.

   (c) Find the x- and y-coordinates of the relative minimum points of f in terms of b.

   (d) Show that for all values of $b > 0$, the relative maximum and minimum points lie on a function of the form $y = -ax^3$ by finding the value of a.

(a) In the viewing window provided below, graph the members of the family $f(x) = x^3 - 3bx$ with $b = 1$, $b = 2$, and $b = 3$. Label each graph.



Viewing Window
$[-5, 5] \times [-15, 15]$

(b) Find the x- and y-coordinates of the relative maximum points of f in terms of b.

(c) Find the x- and y-coordinates of the relative minimum points of f in terms of b.

(d) Show that for all values of $b > 0$, the relative maximum and minimum points lie on a function of the form $y = -ax^3$ by finding the value of a.

Section II Part B: Graphing Calculator MAY **NOT** BE USED.

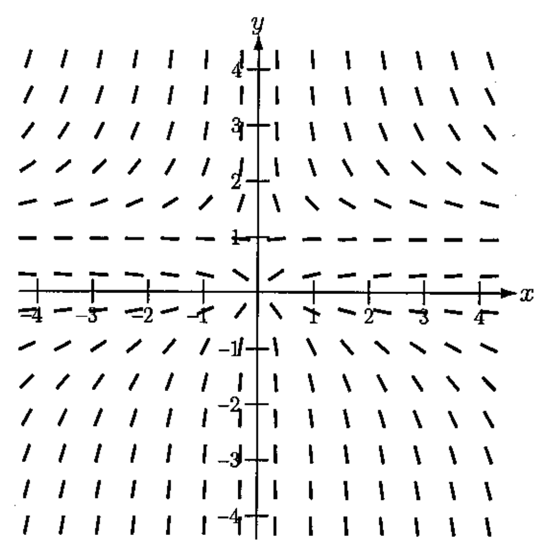
4. Consider the differential equation $\frac{dy}{dx} = \frac{y - y^2}{x}$ for all $x \neq 0$.

(a) Verify that $y = \frac{x}{x+C}$, $x \neq -C$ is a general solution for the given differential equation and show that all solutions contain $(0, 0)$.

(b) Write an equation of the particular solution that contains the point $(1, 2)$, and find the value of $\frac{dy}{dx}$ at $(0, 0)$ for this solution.

(c) Write an equation of the vertical and horizontal asymptotes of the particular solution found in (b).

(d) The slope field for the given differential equation is provided. Sketch both branches of the particular solution curve that passes through the point $(1, 2)$.



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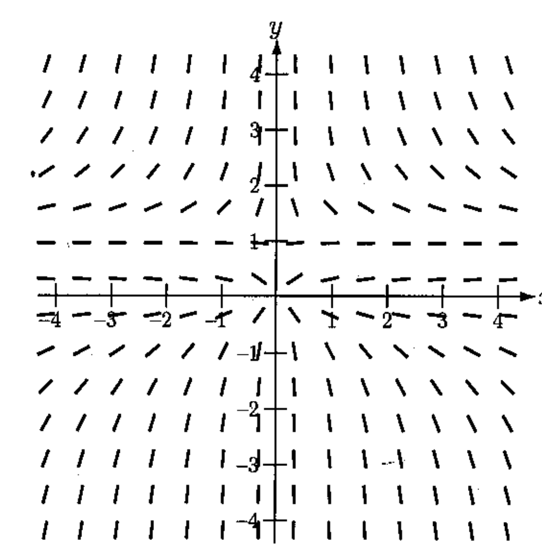

Section II Part B: Graphing Calculator MAY **NOT** BE USED.

5. Consider the relation defined by the equation $\tan y = x + y$ for x in the open interval $-2\pi < x < 2\pi$.

   (a) Find $\frac{dy}{dx}$ in terms of y.

   (b) Find the x- and y-coordinate of each point where the tangent line to the graph is vertical.

   (c) Find $\frac{d^2y}{dx^2}$ in terms of y.

(a) Find $\frac{dy}{dx}$ in terms of y.

(b) Find the x- and y-coordinate of each point where the tangent line to the graph is vertical.

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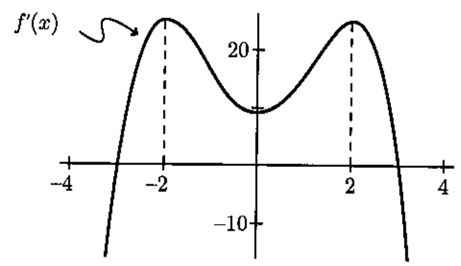
Section II Part B: Graphing Calculator MAY **NOT** BE USED.

6.



The figure above shows the graph of f', the derivative of f. The domain of f is $-4 \le x \le 4$. The derivative of f is an <u>even</u> function and $f'(-3) = f'(3) = 0$.

(a) For what value of x in the interval $-4 \le x \le 4$ does f have a relative maximum? Justify your answer.

(b) For what value of x in the interval $-4 \le x \le 4$ does f have a relative minimum? Justify your answer.

(c) For what values of x is the graph of f concave downward. Use f' to justify your answer.

(d) If $f(0) = 0$, find the value of $\int_{-a}^{a} f(x)\, dx$. Justify your answer.

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