# Pre-Calculus/Trigonometry

73 31. If $\log_a\left(2^a\right)=\frac{a}{4}$, then $a=$

(A) 2 (B) 4 (C) 8 (D) 16 (E) 32

73 2. If $f(x)=x^3+3x^2+4x+5$ and $g(x)=5$, then $g(f(x))=$

(A) $5x^2+15x+25$ (B) $5x^3+15x^2+20x+25$ (C) 1125
(D) 225 (E) 5

88 2. What is the domain of the function $f$ given by $f(x)=\frac{\sqrt{x^2-4}}{x-3}$?

(A) $\{x: x\neq 3\}$ (B) $\{x: |x|\leq 2\}$ (C) $\{x: |x|\geq 2\}$
(D) $\{x: |x|\geq 2 \text{ and } x\neq 3\}$ (E) $\{x: x\geq 2 \text{ and } x\neq 3\}$

88 32. Which of the following does NOT have a period of $\pi$ ?

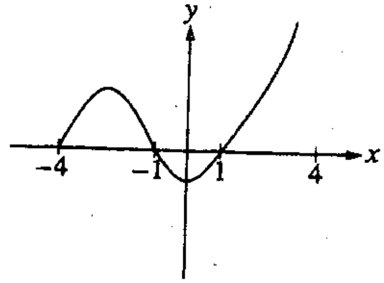
(A) $f(x)=\sin\left(\frac{1}{2}x\right)$ (B) $f(x)=|\sin x|$ (C) $f(x)=\sin^2 x$
(D) $f(x)=\tan x$ (E) $f(x)=\tan^2 x$

73 5. If $f(x)=e^x$, which of the following lines is an asymptote to the graph of $f$?

(A) $y=0$ (B) $x=0$ (C) $y=x$ (D) $y=-x$ (E) $y=1$

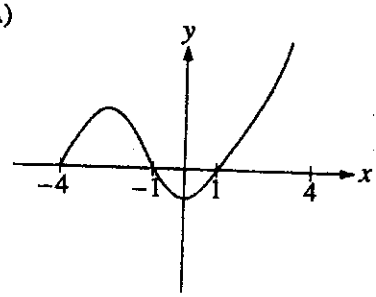
88 42. The graph of which of the following equations has $y=1$ as an asymptote?

(A) $y=\ln x$ (B) $y=\sin x$ (C) $y=\frac{x}{x+1}$ (D) $y=\frac{x^2}{x-1}$ (E) $y=e^{-x}$

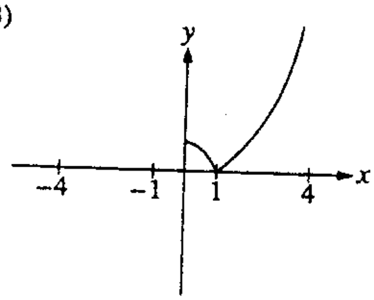


93 40. The graph of $y = f(x)$ is shown in the figure above. Which of the following could be the graph of $y = f(|x|)$?

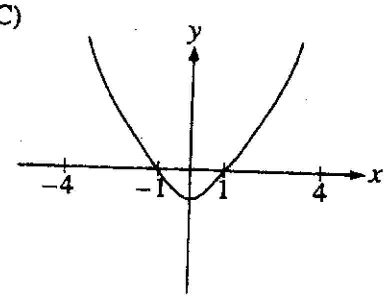
(A)



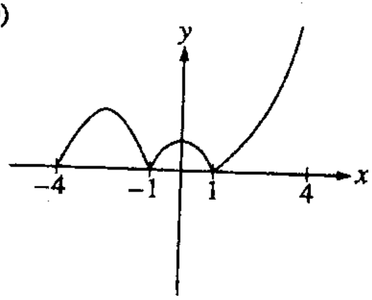
(B)



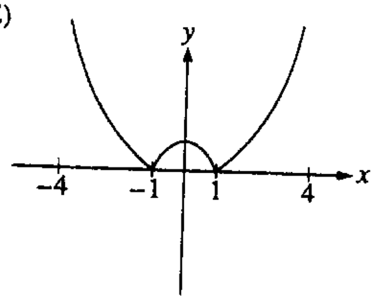
(C)



(D)



(E)



93 5. If the function $f$ is continuous for all real numbers and if $f(x) = \dfrac{x^2 - 4}{x+2}$ when $x \neq -2$, then $f(-2) =$

(A) $-4$ (B) $-2$ (C) $-1$ (D) $0$ (E) $2$

85 29. Which of the following functions are continuous for all real numbers $x$?

I. $y = x^{\frac{2}{3}}$

II. $y = e^x$

III. $y = \tan x$

(A) None (B) I only (C) II only (D) I and II (E) I and III

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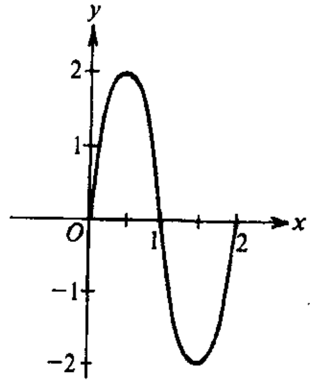


85 35. The figure above shows the graph of a sine function for one complete period. Which of the following is an equation for the graph?

(A) $y = 2\sin\left(\frac{\pi}{2}x\right)$ (B) $y = \sin(\pi x)$ (C) $y = 2\sin(2x)$

(D) $y = 2\sin(\pi x)$ (E) $y = \sin(2x)$

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88 22. If $\ln x - \ln\left(\frac{1}{x}\right) = 2$, then $x =$

(A) $\frac{1}{e^2}$ (B) $\frac{1}{e}$ (C) $e$ (D) $2e$ (E) $e^2$

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93 13. The fundamental period of $2\cos(3x)$ is

(A) $\frac{2\pi}{3}$ (B) $2\pi$ (C) $6\pi$ (D) 2 (E) 3

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69 AB 12. If $f(x) = \frac{4}{x-1}$ and $g(x) = 2x$, then the solution set of $f(g(x)) = g(f(x))$ is

(A) $\left\{\frac{1}{3}\right\}$ (B) $\{2\}$ (C) $\{3\}$ (D) $\{-1, 2\}$ (E) $\left\{\frac{1}{3}, 2\right\}$

# Limits

93 3. $\lim_{n\to\infty} \frac{3n^3-5n}{n^3-2n^2+1}$ is

(A) $-5$ (B) $-2$ (C) 1 (D) 3 (E) nonexistent

93 29. $\lim_{\theta\to 0} \frac{1-\cos\theta}{2\sin^2\theta}$ is

(A) 0 (B) $\frac{1}{8}$ (C) $\frac{1}{4}$ (D) 1 (E) nonexistent

85 37. $\lim_{x\to 0}(x\csc x)$ is

(A) $-\infty$ (B) $-1$ (C) 0 (D) 1 (E) $\infty$

85 5. $\lim_{n\to\infty} \frac{4n^2}{n^2+10{,}000n}$ is

(A) 0 (B) $\frac{1}{2{,}500}$ (C) 1 (D) 4 (E) nonexistent

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73 BC 37. $\lim_{x\to 0} \frac{1-\cos^2(2x)}{x^2} =$

(A) $-2$ (B) 0 (C) 1 (D) 2 (E) 4

98 83. If $a \neq 0$, then $\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4}$ is

(A) $\frac{1}{a^2}$ (B) $\frac{1}{2a^2}$ (C) $\frac{1}{6a^2}$ (D) 0 (E) nonexistent

97 21. $\lim_{x\to 1} \frac{x}{\ln x}$ is

(A) 0 (B) $\frac{1}{e}$ (C) 1 (D) $e$ (E) nonexistent

88 BC 35. If $k$ is a positive integer, then $\lim_{x\to +\infty} \frac{x^k}{e^x}$ is

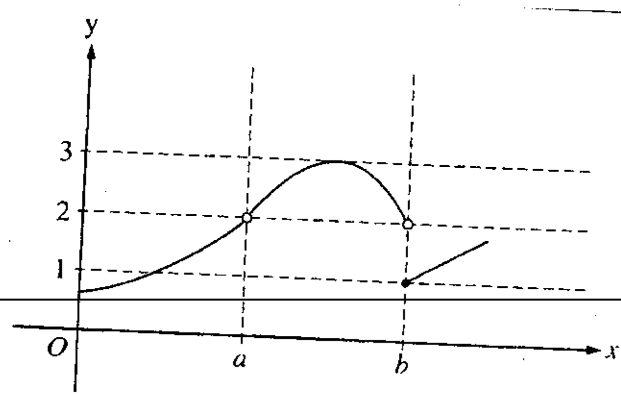
(A) 0 (B) 1 (C) $e$ (D) $k!$ (E) nonexistent

85 BC 38. $\lim_{x\to\infty} \left(1+5e^x\right)^{\frac{1}{x}}$ is

(A) 0 (B) 1 (C) $e$ (D) $e^5$ (E) nonexistent

93 BC 2. If $f(x)=2x^2+1$, then $\lim_{x\to 0} \frac{f(x)-f(0)}{x^2}$ is

(A) 0 (B) 1 (C) 2 (D) 4 (E) nonexistent




97 15. The graph of the function $f$ is shown in the figure above. Which of the following statements about $f$ is true?

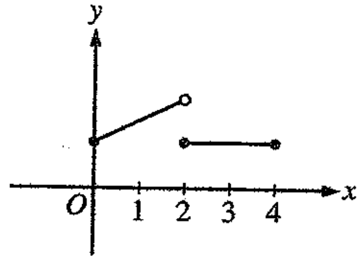
(A) $\lim_{x\to a} f(x) = \lim_{x\to b} f(x)$

(B) $\lim_{x\to a} f(x) = 2$

(C) $\lim_{x\to b} f(x) = 2$

(D) $\lim_{x\to b} f(x) = 1$

(E) $\lim_{x\to a} f(x)$ does not exist.




Graph of $f$

77. The figure above shows the graph of a function $f$ with domain $0 \le x \le 4$. Which of the following statements are true?

I. $\lim_{x \to 2^-} f(x)$ exists.

II. $\lim_{x \to 2^+} f(x)$ exists.

III. $\lim_{x \to 2} f(x)$ exists.

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

97 79. Let $f$ be a function such that $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 5$. Which of the following must be true?

I. $f$ is continuous at $x = 2$.

II. $f$ is differentiable at $x = 2$.

III. The derivative of $f$ is continuous at $x = 2$.

(A) I only (B) II only (C) I and II only (D) I and III only (E) II and III only

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85 41. If $\lim_{x \to a} f(x) = L$, where $L$ is a real number, which of the following must be true?

(A) $f'(a)$ exists.

(B) $f(x)$ is continuous at $x = a$.

(C) $f(x)$ is defined at $x = a$.

(D) $f(a) = L$

(E) None of the above

73 45. If $f$ is a continuous function on $[a,b]$, which of the following is necessarily true?

(A) $f'$ exists on $(a,b)$.

(B) If $f(x_0)$ is a maximum of $f$, then $f'(x_0)=0$.

(C) $\lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right)$ for $x_0 \in (a,b)$

(D) $f'(x)=0$ for some $x \in [a,b]$

(E) The graph of $f'$ is a straight line.

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88 41. If $\lim_{x\to 3} f(x) = 7$, which of the following must be true?

I. $f$ is continuous at $x=3$.
II. $f$ is differentiable at $x=3$.
III. $f(3)=7$

(A) None (B) II only (C) III only

(D) I and III only (E) I, II, and III

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98 12. If $f(x) = \begin{cases} \ln x & \text{for } 0 < x \le 2 \\ x^2 \ln 2 & \text{for } 2 < x \le 4, \end{cases}$ then $\lim_{x\to 2} f(x)$ is

(A) $\ln 2$ (B) $\ln 8$ (C) $\ln 16$ (D) 4 (E) nonexistent

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88 BC 5. Let $f$ be the function defined by the following.

$$f(x) = \begin{cases} \sin x, & x < 0 \\ x^2, & 0 \le x < 1 \\ 2-x, & 1 \le x < 2 \\ x-3, & x \ge 2 \end{cases}$$

For what values of $x$ is $f$ NOT continuous?

(A) 0 only (B) 1 only (C) 2 only (D) 0 and 2 only (E) 0, 1, and 2

88 27. At $x = 3$, the function given by $f(x) = \begin{cases} x^2, & x < 3 \\ 6x - 9, & x \geq 3 \end{cases}$ is

(A) undefined.
(B) continuous but not differentiable.
(C) differentiable but not continuous.
(D) neither continuous nor differentiable.
(E) both continuous and differentiable.

# Limits - BC Level

69 BC 28. What is $\lim_{x \to 0} \frac{e^{2x}-1}{\tan x}$ ?

(A) $-1$ (B) 0 (C) 1 (D) 2 (E) The limit does not exist.

98 BC 79. Let $f$ be the function given by $f(x) = \frac{(x-1)(x^2-4)}{x^2-a}$. For what positive values of $a$ is $f$ continuous for all real numbers $x$?

(A) None
(B) 1 only
(C) 2 only
(D) 4 only
(E) 1 and 4 only

69 AB 3. If $\begin{cases} f(x) = \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, & \text{for } x \neq 2, \\ f(2) = k \end{cases}$ and if $f$ is continuous at $x = 2$, then $k =$

(A) 0 (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) 1 (E) $\frac{7}{5}$

93 BC 42. $\lim_{x \to 0} (1+2x)^{\csc x} =$

(A) 0 (B) 1 (C) 2 (D) $e$ (E) $e^2$

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73 BC 18. Let $g$ be a continuous function on the closed interval $[0,1]$. Let $g(0) = 1$ and $g(1) = 0$. Which of the following is NOT necessarily true?

(A) There exists a number $h$ in $[0,1]$ such that $g(h) \geq g(x)$ for all $x$ in $[0,1]$.

(B) For all $a$ and $b$ in $[0,1]$, if $a = b$, then $g(a) = g(b)$.

(C) There exists a number $h$ in $[0,1]$ such that $g(h) = \frac{1}{2}$.

(D) There exists a number $h$ in $[0,1]$ such that $g(h) = \frac{3}{2}$.

(E) For all $h$ in the open interval $(0,1)$, $\lim_{x \to h} g(x) = g(h)$.

# Definition of a Derivative

85 25. If $f(x)=e^x$, which of the following is equal to $f'(e)$?

(A) $\lim_{h\to 0}\frac{e^{x+h}}{h}$

(B) $\lim_{h\to 0}\frac{e^{x+h}-e^e}{h}$

(C) $\lim_{h\to 0}\frac{e^{e+h}-e}{h}$

(D) $\lim_{h\to 0}\frac{e^{x+h}-1}{h}$

(E) $\lim_{h\to 0}\frac{e^{e+h}-e^e}{h}$

88 29. The $\lim_{h\to 0}\frac{\tan 3(x+h)-\tan 3x}{h}$ is

(A) 0 (B) $3\sec^2(3x)$ (C) $\sec^2(3x)$ (D) $3\cot(3x)$ (E) nonexistent

88 BC 10. $\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$ is

(A) 0 (B) 1 (C) $\sin x$ (D) $\cos x$ (E) nonexistent

69 AB 6. What is $\lim_{h\to 0}\frac{8\left(\frac{1}{2}+h\right)^8-8\left(\frac{1}{2}\right)^8}{h}$?

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) The limit does not exist.

(E) It cannot be determined from the information given.

97 BC 16. $\lim_{h\to 0}\frac{e^h-1}{2h}$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $e$ (E) nonexistent

85 BC 8. If $f$ is a function such that $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=0$, which of the following must be true?

(A) The limit of $f(x)$ as $x$ approaches 2 does not exist.

(B) $f$ is not defined at $x=2$.

(C) The derivative of $f$ at $x=2$ is 0.

(D) $f$ is continuous at $x=0$.

(E) $f(2)=0$

93 37. If $f$ is a differentiable function, then $f'(a)$ is given by which of the following?

I. $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

II. $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

III. $\lim_{x\to a}\frac{f(x+h)-f(x)}{h}$

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

What is $\lim_{h\to 0}\frac{\frac{1}{2}(2+h)^6-\frac{1}{2}(2)^6}{h}$ ?

b. 16 b. 48 c. 64 d. 96 e. none of these

$\lim_{h\to 0}\frac{(2+h)^4-2^4}{h}$

$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{2+h}-\frac{1}{2}\right)$

$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}$

$\lim_{m\to 0}\frac{e^m-1}{m}$

97 BC

78. $\lim_{h \to 0} \frac{\ln(e+h)-1}{h}$ is

(A) $f'(e)$, where $f(x) = \ln x$

(B) $f'(e)$, where $f(x) = \frac{\ln x}{x}$

(C) $f'(1)$, where $f(x) = \ln x$

(D) $f'(1)$, where $f(x) = \ln(x+e)$

(E) $f'(0)$, where $f(x) = \ln x$

# Derivative Rules

85 6. If $f(x) = x$, then $f'(5) =$

(A) 0 (B) $\frac{1}{5}$ (C) 1 (D) 5 (E) $\frac{25}{2}$

97 4. If $f(x) = -x^3 + x + \frac{1}{x}$, then $f'(-1) =$

(A) 3 (B) 1 (C) $-1$ (D) $-3$ (E) $-5$

88 15. If $f(x) = \sqrt{2x}$, then $f'(2) =$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{\sqrt{2}}{2}$ (D) 1 (E) $\sqrt{2}$

93 1. If $f(x) = x^{\frac{3}{2}}$, then $f'(4) =$

(A) $-6$ (B) $-3$ (C) 3 (D) 6 (E) 8

93 24. If $f(x) = (x^2 - 2x - 1)^{\frac{2}{3}}$, then $f'(0)$ is

(A) $\frac{4}{3}$ (B) 0 (C) $-\frac{2}{3}$ (D) $-\frac{4}{3}$ (E) $-2$

85 23. $\frac{d}{dx}\left(\frac{1}{x^3} - \frac{1}{x} + x^2\right)$ at $x = -1$ is

(A) $-6$ (B) $-4$ (C) 0 (D) 2 (E) 6

85 2. If $f(x) = (2x+1)^4$, then the 4th derivative of $f(x)$ at $x = 0$ is

(A) 0 (B) 24 (C) 48 (D) 240 (E) 384

93 6. If $f(x) = \dfrac{x-1}{x+1}$ for all $x \neq -1$, then $f'(1) =$

(A) $-1$  (B) $-\dfrac{1}{2}$  (C) $0$  (D) $\dfrac{1}{2}$  (E) $1$

85 3. If $y = \dfrac{3}{4+x^2}$, then $\dfrac{dy}{dx} =$

(A) $\dfrac{-6x}{\left(4+x^2\right)^2}$  (B) $\dfrac{3x}{\left(4+x^2\right)^2}$  (C) $\dfrac{6x}{\left(4+x^2\right)^2}$  (D) $\dfrac{-3}{\left(4+x^2\right)^2}$  (E) $\dfrac{3}{2x}$

97 2. If $f(x) = x\sqrt{2x-3}$, then $f'(x) =$

(A) $\dfrac{3x-3}{\sqrt{2x-3}}$

(B) $\dfrac{x}{\sqrt{2x-3}}$

(C) $\dfrac{1}{\sqrt{2x-3}}$

(D) $\dfrac{-x+3}{\sqrt{2x-3}}$

(E) $\dfrac{5x-6}{2\sqrt{2x-3}}$

93 10. If $f(x) = (x-1)^2 \sin x$, then $f'(0) =$

(A) $-2$  (B) $-1$  (C) $0$  (D) $1$  (E) $2$

93 8. If $y = \tan x - \cot x$, then $\dfrac{dy}{dx} =$

(A) $\sec x \csc x$  (B) $\sec x - \csc x$  (C) $\sec x + \csc x$  (D) $\sec^2 x - \csc^2 x$  (E) $\sec^2 x + \csc^2 x$

97 7. $\frac{d}{dx}\cos^2(x^3) =$

(A) $6x^2\sin(x^3)\cos(x^3)$

(B) $6x^2\cos(x^3)$

(C) $\sin^2(x^3)$

(D) $-6x^2\sin(x^3)\cos(x^3)$

(E) $-2\sin(x^3)\cos(x^3)$

98 28. If $f(x) = \tan(2x)$, then $f'\left(\frac{\pi}{6}\right) =$

(A) $\sqrt{3}$ (B) $2\sqrt{3}$ (C) 4 (D) $4\sqrt{3}$ (E) 8

18 16. If $f(x) = \sin\left(e^{-x}\right)$, then $f'(x) =$

(A) $-\cos(e^{-x})$

(B) $\cos(e^{-x}) + e^{-x}$

(C) $\cos(e^{-x}) - e^{-x}$

(D) $e^{-x}\cos(e^{-x})$

(E) $-e^{-x}\cos(e^{-x})$

88 12. If $f(x) = \sin x$, then $f'\left(\frac{\pi}{3}\right) =$

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) $\frac{\sqrt{2}}{2}$ (D) $\frac{\sqrt{3}}{2}$ (E) $\sqrt{3}$

8 18. If $y = 2\cos\left(\frac{x}{2}\right)$, then $\frac{d^2y}{dx^2} =$

(A) $-8\cos\left(\frac{x}{2}\right)$ (B) $-2\cos\left(\frac{x}{2}\right)$ (C) $-\sin\left(\frac{x}{2}\right)$ (D) $-\cos\left(\frac{x}{2}\right)$ (E) $-\frac{1}{2}\cos\left(\frac{x}{2}\right)$

73 4. If $f(x)=x+\sin x$, then $f'(x)=$

(A) $1+\cos x$ (B) $1-\cos x$ (C) $\cos x$

(D) $\sin x - x\cos x$ (E) $\sin x + x\cos x$

73 9. If $y=\cos^2 3x$, then $\frac{dy}{dx}=$

(A) $-6\sin 3x\cos 3x$ (B) $-2\cos 3x$ (C) $2\cos 3x$

(D) $6\cos 3x$ (E) $2\sin 3x\cos 3x$

85 18. If $y=\cos^2 x-\sin^2 x$, then $y'=$

(A) $-1$ (B) $0$ (C) $-2\sin(2x)$ (D) $-2(\cos x+\sin x)$ (E) $2(\cos x-\sin x)$

85 BC 6. If $f(x)=\frac{x}{\tan x}$, then $f'\left(\frac{\pi}{4}\right)=$

(A) $2$ (B) $\frac{1}{2}$ (C) $1+\frac{\pi}{2}$ (D) $\frac{\pi}{2}-1$ (E) $1-\frac{\pi}{2}$

93 31. If $f(x)=e^{3\ln(x^2)}$, then $f'(x)=$

(A) $e^{3\ln(x^2)}$ (B) $\frac{3}{x^2}e^{3\ln(x^2)}$ (C) $6(\ln x)e^{3\ln(x^2)}$ (D) $5x^4$ (E) $6x^5$

93 BC 15. If $f(x)=e^{\tan^2 x}$, then $f'(x)=$

(A) $e^{\tan^2 x}$

(B) $\sec^2 x\, e^{\tan^2 x}$

(C) $\tan^2 x\, e^{\tan^2 x-1}$

(D) $2\tan x\sec^2 x\, e^{\tan^2 x}$

(E) $2\tan x\, e^{\tan^2 x}$

93 BC 8. If $f(x)=\ln(e^{2x})$, then $f'(x)=$

(A) 1 (B) 2 (C) $2x$ (D) $e^{-2x}$ (E) $2e^{-2x}$

97 76. If $f(x)=\dfrac{e^{2x}}{2x}$, then $f'(x)=$

(A) 1

(B) $\dfrac{e^{2x}(1-2x)}{2x^2}$

(C) $e^{2x}$

(D) $\dfrac{e^{2x}(2x+1)}{x^2}$

(E) $\dfrac{e^{2x}(2x-1)}{2x^2}$

97 BC 5. If $f(x)=(x-1)^{\frac{3}{2}}+\dfrac{e^{x-2}}{2}$, then $f'(2)=$

(A) 1 (B) $\dfrac{3}{2}$ (C) 2 (D) $\dfrac{7}{2}$ (E) $\dfrac{3+e}{2}$

69 AB 22. $\dfrac{d}{dx}\left(\ln e^{2x}\right)=$

(A) $\dfrac{1}{e^{2x}}$ (B) $\dfrac{2}{e^{2x}}$ (C) $2x$ (D) 1 (E) 2

85 BC 11. $\dfrac{d}{dx}\ln\left(\dfrac{1}{1-x}\right)=$

(A) $\dfrac{1}{1-x}$ (B) $\dfrac{1}{x-1}$ (C) $1-x$ (D) $x-1$ (E) $(1-x)^2$

88 1. If $y = x^2 e^x$, then $\frac{dy}{dx} =$

(A) $2xe^x$ (B) $x\left(x+2e^x\right)$ (C) $xe^x(x+2)$

(D) $2x+e^x$ (E) $2x+e$

88 6. If $y = \frac{\ln x}{x}$, then $\frac{dy}{dx} =$

(A) $\frac{1}{x}$ (B) $\frac{1}{x^2}$ (C) $\frac{\ln x - 1}{x^2}$ (D) $\frac{1-\ln x}{x^2}$ (E) $\frac{1+\ln x}{x^2}$

88 BC 3. If $f(x) = \ln\left(\sqrt{x}\right)$, then $f''(x) =$

(A) $-\frac{2}{x^2}$ (B) $-\frac{1}{2x^2}$ (C) $-\frac{1}{2x}$ (D) $-\frac{1}{2x^{\frac{3}{2}}}$ (E) $\frac{2}{x^2}$

85 BC 17. If $f(x) = x\ln\left(x^2\right)$, then $f'(x) =$

(A) $\ln\left(x^2\right)+1$ (B) $\ln\left(x^2\right)+2$ (C) $\ln\left(x^2\right)+\frac{1}{x}$ (D) $\frac{1}{x^2}$ (E) $\frac{1}{x}$

88 BC 8. If $f(x) = e^x$, then $\ln\left(f'(2)\right) =$

(A) 2 (B) 0 (C) $\frac{1}{e^2}$ (D) $2e$ (E) $e^2$

73 BC 1. If $f(x) = e^{1/x}$, then $f'(x) =$

(A) $-\frac{e^{1/x}}{x^2}$ (B) $-e^{1/x}$ (C) $\frac{e^{1/x}}{x}$ (D) $\frac{e^{1/x}}{x^2}$ (E) $\frac{1}{x}e^{(1/x)-1}$

73 BC 31. If $f(x) = \ln(\ln x)$, then $f'(x) =$

(A) $\frac{1}{x}$ (B) $\frac{1}{\ln x}$ (C) $\frac{\ln x}{x}$ (D) $x$ (E) $\frac{1}{x\ln x}$

93 25. $\frac{d}{dx}\left(2^x\right)=$

(A) $2^{x-1}$ (B) $(2^{x-1})x$ (C) $(2^x)\ln 2$ (D) $(2^{x-1})\ln 2$ (E) $\frac{2x}{\ln 2}$

85 10. If $y=10^{(x^2-1)}$, then $\frac{dy}{dx}=$

(A) $(\ln 10)10^{(x^2-1)}$ (B) $(2x)10^{(x^2-1)}$ (C) $(x^2-1)10^{(x^2-2)}$

(D) $2x(\ln 10)10^{(x^2-1)}$ (E) $x^2(\ln 10)10^{(x^2-1)}$

8 10. What is the instantaneous rate of change at $x=2$ of the function $f$ given by $f(x)=\frac{x^2-2}{x-1}$ ?

(A) $-2$ (B) $\frac{1}{6}$ (C) $\frac{1}{2}$ (D) 2 (E) 6

93 BC 21. The value of the derivative of $y=\frac{\sqrt[3]{x^2+8}}{\sqrt[4]{2x+1}}$ at $x=0$ is

(A) $-1$ (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

88 BC 4. If $u$, $v$, and $w$ are nonzero differentiable functions, then the derivative of $\frac{uv}{w}$ is

(A) $\frac{uv'+u'v}{w'}$ (B) $\frac{u'v'w-uvw'}{w^2}$ (C) $\frac{uvw'-uv'w-u'vw}{w^2}$

(D) $\frac{u'vw+uv'w+uvw'}{w^2}$ (E) $\frac{uv'w+u'vw-uvw'}{w^2}$

69 39. If $y=\tan u$, $u=v-\frac{1}{v}$, and $v=\ln x$, what is the value of $\frac{dy}{dx}$ at $x=e$?

(A) 0 (B) $\frac{1}{e}$ (C) 1 (D) $\frac{2}{e}$ (E) $\sec^2 e$

98 8. Let $f$ and $g$ be differentiable functions with the following properties:

(i) $g(x) > 0$ for all $x$
(ii) $f(0) = 1$

If $h(x) = f(x)g(x)$ and $h'(x) = f(x)g'(x)$, then $f(x) =$

(A) $f'(x)$ (B) $g(x)$ (C) $e^x$ (D) 0 (E) 1

98 BC 5. If $f$ and $g$ are twice differentiable and if $h(x) = f(g(x))$, then $h''(x) =$

(A) $f''(g(x))[g'(x)]^2 + f'(g(x))g''(x)$

(B) $f''(g(x))g'(x) + f'(g(x))g''(x)$

(C) $f''(g(x))[g'(x)]^2$

(D) $f''(g(x))g''(x)$

(E) $f''(g(x))$

73 18. $\dfrac{d}{dx}(\arcsin 2x) =$

(A) $\dfrac{-1}{2\sqrt{1-4x^2}}$ (B) $\dfrac{-2}{\sqrt{4x^2-1}}$ (C) $\dfrac{1}{2\sqrt{1-4x^2}}$

(D) $\dfrac{2}{\sqrt{1-4x^2}}$ (E) $\dfrac{2}{\sqrt{4x^2-1}}$

85 20. If $y = \arctan(\cos x)$, then $\dfrac{dy}{dx} =$

(A) $\dfrac{-\sin x}{1+\cos^2 x}$ (B) $-(\operatorname{arcsec}(\cos x))^2 \sin x$ (C) $(\operatorname{arcsec}(\cos x))^2$

(D) $\dfrac{1}{(\arccos x)^2+1}$ (E) $\dfrac{1}{1+\cos^2 x}$

73 BC

39. Let $f$ and $g$ be differentiable functions such that

$f(1) = 2$, $\quad f'(1) = 3$, $\quad f'(2) = -4$,

$g(1) = 2$, $\quad g'(1) = -3$, $\quad g'(2) = 5$.

If $h(x) = f\left(g(x)\right)$, then $h'(1) =$

(A) $-9$ (B) $-4$ (C) 0 (D) 12 (E) 15

# Derivatives – BC Level (or no longer tested on AB)

88 BC 22. If $f(x)=\left(x^2+1\right)^x$, then $f'(x)=$

(A) $x\left(x^2+1\right)^{x-1}$

(B) $2x^2\left(x^2+1\right)^{x-1}$

(C) $x\ln\left(x^2+1\right)$

(D) $\ln\left(x^2+1\right)+\dfrac{2x^2}{x^2+1}$

(E) $\left(x^2+1\right)^x\left[\ln\left(x^2+1\right)+\dfrac{2x^2}{x^2+1}\right]$

---

85 BC 26. For $0<x<\dfrac{\pi}{2}$, if $y=(\sin x)^x$, then $\dfrac{dy}{dx}$ is

(A) $x\ln(\sin x)$ (B) $(\sin x)^x\cot x$ (C) $x(\sin x)^{x-1}(\cos x)$

(D) $(\sin x)^x(x\cos x+\sin x)$ (E) $(\sin x)^x\left(x\cot x+\ln(\sin x)\right)$

---

88 24. $\dfrac{d}{dx}\left(x^{\ln x}\right)=$

(A) $x^{\ln x}$ (B) $(\ln x)^x$ (C) $\dfrac{2}{x}(\ln x)\left(x^{\ln x}\right)$ (D) $(\ln x)\left(x^{\ln x-1}\right)$ (E) $2(\ln x)\left(x^{\ln x}\right)$

---

69 BC 38. If $f(x)=\left(x^2+1\right)^{(2-3x)}$, then $f'(1)=$

(A) $-\dfrac{1}{2}\ln(8e)$ (B) $-\ln(8e)$ (C) $-\dfrac{3}{2}\ln(2)$ (D) $-\dfrac{1}{2}$ (E) $\dfrac{1}{8}$

73 BC

32. If $y = x^{\ln x}$, then $y'$ is

(A) $\dfrac{x^{\ln x} \ln x}{x^2}$

(B) $x^{1/x} \ln x$

(C) $\dfrac{2x^{\ln x} \ln x}{x}$

(D) $\dfrac{x^{\ln x} \ln x}{x}$

(E) None of the above

97 BC

4. $\dfrac{d}{dx}\left(xe^{\ln x^2}\right) =$

(A) $1+2x$ (B) $x+x^2$ (C) $3x^2$ (D) $x^3$ (E) $x^2+x^3$

93 BC

18. If $e^{f(x)} = 1+x^2$, then $f'(x) =$

(A) $\dfrac{1}{1+x^2}$ (B) $\dfrac{2x}{1+x^2}$ (C) $2x(1+x^2)$ (D) $2x\left(e^{1+x^2}\right)$ (E) $2x\ln(1+x^2)$

88 BC

28. $\dfrac{d}{dx}\ln\left|\cos\left(\dfrac{\pi}{x}\right)\right|$ is

(A) $\dfrac{-\pi}{x^2\cos\left(\dfrac{\pi}{x}\right)}$ (B) $-\tan\left(\dfrac{\pi}{x}\right)$ (C) $\dfrac{1}{\cos\left(\dfrac{\pi}{x}\right)}$

(D) $\dfrac{\pi}{x}\tan\left(\dfrac{\pi}{x}\right)$ (E) $\dfrac{\pi}{x^2}\tan\left(\dfrac{\pi}{x}\right)$

69 AB

18. If $f(x) = 2+|x-3|$ for all $x$, then the value of the derivative $f'(x)$ at $x=3$ is

(A) $-1$ (B) $0$ (C) $1$ (D) $2$ (E) nonexistent

97 19. If $f(x) = \ln\left|x^2 - 1\right|$, then $f'(x) =$

(A) $\left|\dfrac{2x}{x^2-1}\right|$

(B) $\dfrac{2x}{\left|x^2-1\right|}$

(C) $\dfrac{2|x|}{x^2-1}$

(D) $\dfrac{2x}{x^2-1}$

(E) $\dfrac{1}{x^2-1}$

---

85 BC 30. If $x = t^3 - t$ and $y = \sqrt{3t+1}$, then $\dfrac{dy}{dx}$ at $t = 1$ is

(A) $\dfrac{1}{8}$ (B) $\dfrac{3}{8}$ (C) $\dfrac{3}{4}$ (D) $\dfrac{8}{3}$ (E) 8

---

69 BC 14. If $y = x^2 + 2$ and $u = 2x - 1$, then $\dfrac{dy}{du} =$

(A) $\dfrac{2x^2-2x+4}{(2x-1)^2}$ (B) $6x^2 - 2x + 4$ (C) $x^2$

(D) $x$ (E) $\dfrac{1}{x}$

---

97 BC 2. If $x = e^{2t}$ and $y = \sin(2t)$, then $\dfrac{dy}{dx} =$

(A) $4e^{2t}\cos(2t)$ (B) $\dfrac{e^{2t}}{\cos(2t)}$ (C) $\dfrac{\sin(2t)}{2e^{2t}}$ (D) $\dfrac{\cos(2t)}{2e^{2t}}$ (E) $\dfrac{\cos(2t)}{e^{2t}}$

93 BC 6. If $x = t^2 + 1$ and $y = t^3$, then $\frac{d^2y}{dx^2} =$

(A) $\frac{3}{4t}$ (B) $\frac{3}{2t}$ (C) $3t$ (D) $6t$ (E) $\frac{3}{2}$

85 BC 19. If $f$ and $g$ are twice differentiable functions such that $g(x) = e^{f(x)}$ and $g''(x) = h(x)e^{f(x)}$, then $h(x) =$

(A) $f'(x) + f''(x)$ (B) $f'(x) + (f''(x))^2$ (C) $(f'(x) + f''(x))^2$

(D) $(f'(x))^2 + f''(x)$ (E) $2f'(x) + f''(x)$

93 BC 24. Let $f$ and $g$ be functions that are differentiable for all real numbers, with $g(x) \neq 0$ for $x \neq 0$. If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to 0} \frac{f(x)}{g(x)}$ is

(A) 0

(B) $\frac{f'(x)}{g'(x)}$

(C) $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$

(D) $\frac{f'(x)g(x) - f(x)g'(x)}{(f(x))^2}$

(E) nonexistent

69 45. If $\frac{d}{dx}(f(x)) = g(x)$ and $\frac{d}{dx}(g(x)) = f(x^2)$, then $\frac{d^2}{dx^2}\left(f(x^3)\right) =$

(A) $f\left(x^6\right)$ (B) $g\left(x^3\right)$ (C) $3x^2 g\left(x^3\right)$

(D) $9x^4 f\left(x^6\right) + 6x\,g\left(x^3\right)$ (E) $f\left(x^6\right) + g\left(x^3\right)$

93 BC 26. If $y = \arctan(e^{2x})$, then $\frac{dy}{dx} =$

(A) $\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$ (B) $\frac{2e^{2x}}{1+e^{4x}}$ (C) $\frac{e^{2x}}{1+e^{4x}}$ (D) $\frac{1}{\sqrt{1-e^{4x}}}$ (E) $\frac{1}{1+e^{4x}}$

# Derivatives - Tangent & Normal Lines

73 3. The slope of the line tangent to the graph of $y = \ln\left(x^2\right)$ at $x = e^2$ is

(A) $\frac{1}{e^2}$ (B) $\frac{2}{e^2}$ (C) $\frac{4}{e^2}$ (D) $\frac{1}{e^4}$ (E) $\frac{4}{e^4}$

98 18. An equation of the line tangent to the graph of $y = x + \cos x$ at the point $(0,1)$ is

(A) $y = 2x + 1$ (B) $y = x + 1$ (C) $y = x$ (D) $y = x - 1$ (E) $y = 0$

93 7. An equation of the line tangent to the graph of $y = \frac{2x+3}{3x-2}$ at the point $(1,5)$ is

(A) $13x - y = 8$ (B) $13x + y = 18$ (C) $x - 13y = 64$
(D) $x + 13y = 66$ (E) $-2x + 3y = 13$

88 11. An equation of the line tangent to the graph of $f(x) = x(1-2x)^3$ at the point $(1,-1)$ is

(A) $y = -7x + 6$ (B) $y = -6x + 5$ (C) $y = -2x + 1$
(D) $y = 2x - 3$ (E) $y = 7x - 8$

97 12. At what point on the graph of $y = \frac{1}{2}x^2$ is the tangent line parallel to the line $2x - 4y = 3$?

(A) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ (B) $\left(\frac{1}{2}, \frac{1}{8}\right)$ (C) $\left(1, -\frac{1}{4}\right)$ (D) $\left(1, \frac{1}{2}\right)$ (E) $(2,2)$

73 11. If the line $3x - 4y = 0$ is tangent in the first quadrant to the curve $y = x^3 + k$, then $k$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $0$ (D) $-\frac{1}{8}$ (E) $-\frac{1}{2}$

69 AB 20. An equation for a tangent to the graph of $y = \arcsin\frac{x}{2}$ at the origin is

(A) $x - 2y = 0$ (B) $x - y = 0$ (C) $x = 0$ (D) $y = 0$ (E) $\pi x - 2y = 0$

---

85 8. The slope of the line tangent to the graph of $y = \ln\left(\frac{x}{2}\right)$ at $x = 4$ is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1 (E) 4

---

97 10. An equation of the line tangent to the graph of $y = \cos(2x)$ at $x = \frac{\pi}{4}$ is

(A) $y - 1 = -\left(x - \frac{\pi}{4}\right)$

(B) $y - 1 = -2\left(x - \frac{\pi}{4}\right)$

(C) $y = 2\left(x - \frac{\pi}{4}\right)$

(D) $y = -\left(x - \frac{\pi}{4}\right)$

(E) $y = -2\left(x - \frac{\pi}{4}\right)$

---

85 43. An equation of the line tangent to $y = x^3 + 3x^2 + 2$ at its point of inflection is

(A) $y = -6x - 6$ (B) $y = -3x + 1$ (C) $y = 2x + 10$

(D) $y = 3x - 1$ (E) $y = 4x + 1$

---

98 87. Which of the following is an equation of the line tangent to the graph of $f(x) = x^4 + 2x^2$ at the point where $f'(x) = 1$?

(A) $y = 8x - 5$
(B) $y = x + 7$
(C) $y = x + 0.763$
(D) $y = x - 0.122$
(E) $y = x - 2.146$

97 80. Let $f$ be the function given by $f(x) = 2e^{4x^2}$. For what value of $x$ is the slope of the line tangent to the graph of $f$ at $(x, f(x))$ equal to 3?

(A) 0.168 (B) 0.276 (C) 0.318 (D) 0.342 (E) 0.551

93 16. The slope of the line normal to the graph of $y = 2\ln(\sec x)$ at $x = \frac{\pi}{4}$ is

(A) $-2$

(B) $-\frac{1}{2}$

(C) $\frac{1}{2}$

(D) 2

(E) nonexistent

85 BC 44. At each point $(x, y)$ on a certain curve, the slope of the curve is $3x^2y$. If the curve contains the point $(0, 8)$, then its equation is

(A) $y = 8e^{x^3}$ (B) $y = x^3 + 8$ (C) $y = e^{x^3} + 7$

(D) $y = \ln(x+1) + 8$ (E) $y^2 = x^3 + 8$

85 BC 32. An equation of the line normal to the graph of $y = x^3 + 3x^2 + 7x - 1$ at the point where $x = -1$ is

(A) $4x + y = -10$ (B) $x - 4y = 23$ (C) $4x - y = 2$ (D) $x + 4y = 25$ (E) $x + 4y = -25$

97 BC 6. The line normal to the curve $y = \sqrt{16 - x}$ at the point $(0, 4)$ has slope

(A) 8 (B) 4 (C) $\frac{1}{8}$ (D) $-\frac{1}{8}$ (E) $-8$

88 BC 11. If $x+7y=29$ is an equation of the line <u>normal</u> to the graph of $f$ at the point $(1,4)$, then $f'(1)=$

(A) 7 (B) $\frac{1}{7}$ (C) $-\frac{1}{7}$ (D) $-\frac{7}{29}$ (E) $-7$

73 BC 4. For what non-negative value of $b$ is the line given by $y=-\frac{1}{3}x+b$ normal to the curve $y=x^3$ ?

(A) 0 (B) 1 (C) $\frac{4}{3}$ (D) $\frac{10}{3}$ (E) $\frac{10\sqrt{3}}{3}$

98 77. Let $f$ be the function given by $f(x)=3e^{2x}$ and let $g$ be the function given by $g(x)=6x^3$. At what value of $x$ do the graphs of $f$ and $g$ have parallel tangent lines?

(A) $-0.701$
(B) $-0.567$
(C) $-0.391$
(D) $-0.302$
(E) $-0.258$

# MVT, IVT, and Other Theorems

85 BC 13. Let $f$ be the function given by $f(x)=x^3-3x^2$. What are all values of $c$ that satisfy the conclusion of the Mean Value Theorem of differential calculus on the closed interval $[0,3]$?

(A) 0 only (B) 2 only (C) 3 only (D) 0 and 3 (E) 2 and 3

88 13. If the function $f$ has a continuous derivative on $[0,c]$, then $\int_0^c f'(x)\,dx =$

(A) $f(c)-f(0)$ (B) $|f(c)-f(0)|$ (C) $f(c)$ (D) $f(x)+c$ (E) $f''(c)-f''(0)$

93 18. If $f(x)=\sin\left(\frac{x}{2}\right)$, then there exists a number $c$ in the interval $\frac{\pi}{2}<x<\frac{3\pi}{2}$ that satisfies the conclusion of the Mean Value Theorem. Which of the following could be $c$?

(A) $\frac{2\pi}{3}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{6}$ (D) $\pi$ (E) $\frac{3\pi}{2}$

69 BC 3. The Mean Value Theorem guarantees the existence of a special point on the graph of $y=\sqrt{x}$ between $(0,0)$ and $(4,2)$. What are the coordinates of this point?

(A) $(2,1)$
(B) $(1,1)$
(C) $(2,\sqrt{2})$
(D) $\left(\frac{1}{2},\frac{1}{\sqrt{2}}\right)$
(E) None of the above

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(x)$ | 1 | $k$ | 2 |

98 26. The function $f$ is continuous on the closed interval $[0,2]$ and has values that are given in the table above. The equation $f(x)=\frac{1}{2}$ must have at least two solutions in the interval $[0,2]$ if $k=$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 3

97 BC 81. Let $f$ be a continuous function on the closed interval $[-3,6]$. If $f(-3)=-1$ and $f(6)=3$, then the Intermediate Value Theorem guarantees that

(A) $f(0)=0$

(B) $f'(c)=\frac{4}{9}$ for at least one $c$ between $-3$ and 6

(C) $-1 \le f(x) \le 3$ for all $x$ between $-3$ and 6

(D) $f(c)=1$ for at least one $c$ between $-3$ and 6

(E) $f(c)=0$ for at least one $c$ between $-1$ and 3

85 BC 16. Which of the following functions shows that the statement "If a function is continuous at $x=0$, then it is differentiable at $x=0$" is false?

(A) $f(x)=x^{-\frac{4}{3}}$ (B) $f(x)=x^{-\frac{1}{3}}$ (C) $f(x)=x^{\frac{1}{3}}$ (D) $f(x)=x^{\frac{4}{3}}$ (E) $f(x)=x^3$

98 4. If $f$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$, which of the following could be false?

(A) $f'(c)=\frac{f(b)-f(a)}{b-a}$ for some $c$ such that $a<c<b$.

(B) $f'(c)=0$ for some $c$ such that $a<c<b$.

(C) $f$ has a minimum value on $a \le x \le b$.

(D) $f$ has a maximum value on $a \le x \le b$.

(E) $\int_a^b f(x)dx$ exists.

98 91. Let $f$ be a function that is differentiable on the open interval $(1,10)$. If $f(2)=-5$, $f(5)=5$, and $f(9)=-5$, which of the following must be true?

I. $f$ has at least 2 zeros.
II. The graph of $f$ has at least one horizontal tangent.
III. For some $c$, $2<c<5$, $f(c)=3$.

(A) None
(B) I only
(C) I and II only
(D) I and III only
(E) I, II, and III

# Implicit Differentiation

85 BC 9. If $xy^2 + 2xy = 8$, then, at the point $(1, 2)$, $y'$ is

(A) $-\frac{5}{2}$ (B) $-\frac{4}{3}$ (C) $-1$ (D) $-\frac{1}{2}$ (E) $0$

98 BC 3. The slope of the line tangent to the curve $y^2 + (xy+1)^3 = 0$ at $(2, -1)$ is

(A) $-\frac{3}{2}$ (B) $-\frac{3}{4}$ (C) $0$ (D) $\frac{3}{4}$ (E) $\frac{3}{2}$

69 BC 5. If $3x^2 + 2xy + y^2 = 2$, then the value of $\frac{dy}{dx}$ at $x = 1$ is

(A) $-2$ (B) $0$ (C) $2$ (D) $4$ (E) not defined

97 17. If $x^2 + y^2 = 25$, what is the value of $\frac{d^2y}{dx^2}$ at the point $(4, 3)$?

(A) $-\frac{25}{27}$ (B) $-\frac{7}{27}$ (C) $\frac{7}{27}$ (D) $\frac{3}{4}$ (E) $\frac{25}{27}$

97 BC 10. If $y = xy + x^2 + 1$, then when $x = -1$, $\frac{dy}{dx}$ is

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $-1$ (D) $-2$ (E) nonexistent

93 BC 17. The slope of the line tangent to the graph of $\ln(xy) = x$ at the point where $x = 1$ is

(A) $0$ (B) $1$ (C) $e$ (D) $e^2$ (E) $1 - e$

69 BC 24. If $\sin x = e^y$, $0 < x < \pi$, what is $\frac{dy}{dx}$ in terms of $x$ ?

(A) $-\tan x$ (B) $-\cot x$ (C) $\cot x$ (D) $\tan x$ (E) $\csc x$

---

93 4. If $x^3 + 3xy + 2y^3 = 17$, then in terms of $x$ and $y$, $\frac{dy}{dx} =$

(A) $-\frac{x^2 + y}{x + 2y^2}$

(B) $-\frac{x^2 + y}{x + y^2}$

(C) $-\frac{x^2 + y}{x + 2y}$

(D) $-\frac{x^2 + y}{2y^2}$

(E) $\frac{-x^2}{1 + 2y^2}$

---

73 40. If $\tan(xy) = x$, then $\frac{dy}{dx} =$

(A) $\frac{1 - y\tan(xy)\sec(xy)}{x\tan(xy)\sec(xy)}$ (B) $\frac{\sec^2(xy) - y}{x}$ (C) $\cos^2(xy)$

(D) $\frac{\cos^2(xy)}{x}$ (E) $\frac{\cos^2(xy) - y}{x}$

---

88 BC 6. If $y^2 - 2xy = 16$, then $\frac{dy}{dx} =$

(A) $\frac{x}{y - x}$ (B) $\frac{y}{x - y}$ (C) $\frac{y}{y - x}$ (D) $\frac{y}{2y - x}$ (E) $\frac{2y}{x - y}$

88 9. If $x+2xy-y^2=2$, then at the point $(1,1)$, $\frac{dy}{dx}$ is

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$ (C) 0 (D) $-\frac{3}{2}$ (E) nonexistent

98 6. If $x^2+xy=10$, then when $x=2$, $\frac{dy}{dx}=$

(A) $-\frac{7}{2}$ (B) $-2$ (C) $\frac{2}{7}$ (D) $\frac{3}{2}$ (E) $\frac{7}{2}$

85 13. If $x^2+xy+y^3=0$, then, in terms of $x$ and $y$, $\frac{dy}{dx}=$

(A) $-\frac{2x+y}{x+3y^2}$ (B) $-\frac{x+3y^2}{2x+y}$ (C) $\frac{-2x}{1+3y^2}$ (D) $\frac{-2x}{x+3y^2}$ (E) $-\frac{2x+y}{x+3y^2-1}$

8-B

6. Consider the closed curve in the $xy$-plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

(a) Show that $\frac{dy}{dx} = \frac{-(x+1)}{2(y^3+1)}$.

(b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.

(c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.

(d) Is it possible for this curve to have a horizontal tangent at points where it intersects the $x$-axis? Explain your reasoning.

5-B

5. Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

(b) Find all points $(x, y)$ on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

(c) Show that there are no points $(x, y)$ on the curve where the line tangent to the curve is horizontal.

(d) Let $x$ and $y$ be functions of time $t$ that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of $y$ is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$.

4-A

4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point $P$ with $x$-coordinate 3 at which the line tangent to the curve at $P$ is horizontal. Find the $y$-coordinate of $P$.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point $P$ found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point $P$ ? Justify your answer.

00

5. Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose $x$-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the $x$-coordinate of each point on the curve where the tangent line is vertical.

98

6. Consider the curve defined by $2y^3 + 6x^2y - 12x^2 + 6y = 1$.

(a) Show that $\frac{dy}{dx} = \frac{4x - 2xy}{x^2 + y^2 + 1}$.

(b) Write an equation of each horizontal tangent line to the curve.

(c) The line through the origin with slope $-1$ is tangent to the curve at point $P$. Find the $x$– and $y$–coordinates of point $P$.



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### Question 6

Consider the closed curve in the $xy$-plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

(a) Show that $\frac{dy}{dx} = \frac{-(x+1)}{2(y^3+1)}$.

(b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.

(c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.

(d) Is it possible for this curve to have a horizontal tangent at points where it intersects the $x$-axis? Explain your reasoning.

---

| | |
|---|---|
| (a) $2x + 2 + 4y^3 \frac{dy}{dx} + 4\frac{dy}{dx} = 0$<br>$\left(4y^3 + 4\right)\frac{dy}{dx} = -2x - 2$<br>$\frac{dy}{dx} = \frac{-2(x+1)}{4\left(y^3+1\right)} = \frac{-(x+1)}{2\left(y^3+1\right)}$ | 2 : { 1 : implicit differentiation<br>1 : verification |
| (b) $\left.\frac{dy}{dx}\right\rvert_{(-2,\,1)} = \frac{-(-2+1)}{2(1+1)} = \frac{1}{4}$<br>Tangent line: $y = 1 + \frac{1}{4}(x+2)$ | 2 : { 1 : slope<br>1 : tangent line equation |
| (c) Vertical tangent lines occur at points on the curve where $y^3 + 1 = 0$ (or $y = -1$) and $x \neq -1$.<br>On the curve, $y = -1$ implies that $x^2 + 2x + 1 - 4 = 5$, so $x = -4$ or $x = 2$.<br>Vertical tangent lines occur at the points $(-4, -1)$ and $(2, -1)$. | 3 : { 1 : $y = -1$<br>1 : substitutes $y = -1$ into the equation of the curve<br>1 : answer |
| (d) Horizontal tangents occur at points on the curve where $x = -1$ and $y \neq -1$.<br>The curve crosses the $x$-axis where $y = 0$.<br>$(-1)^2 + 2(-1) + 0^4 + 4 \cdot 0 \neq 5$<br>No, the curve cannot have a horizontal tangent where it crosses the $x$-axis. | 2 : { 1 : works with $x = -1$ or $y = 0$<br>1 : answer with reason |

## Question 5

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

(b) Find all points $(x, y)$ on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

(c) Show that there are no points $(x, y)$ on the curve where the line tangent to the curve is horizontal.

(d) Let $x$ and $y$ be functions of time $t$ that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of $y$ is 3 and $\dfrac{dy}{dt} = 6$. Find the value of $\dfrac{dx}{dt}$ at time $t = 5$.

| | |
|---|---|
| (a) $2yy' = y + xy'$<br>$(2y - x)y' = y$<br>$y' = \dfrac{y}{2y - x}$ | $2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$ |
| (b) $\dfrac{y}{2y - x} = \dfrac{1}{2}$<br>$2y = 2y - x$<br>$x = 0$<br>$y = \pm\sqrt{2}$<br>$(0, \sqrt{2}), (0, -\sqrt{2})$ | $2 : \begin{cases} 1 : \dfrac{y}{2y - x} = \dfrac{1}{2} \\ 1 : \text{answer} \end{cases}$ |
| (c) $\dfrac{y}{2y - x} = 0$<br>$y = 0$<br>The curve has no horizontal tangent since<br>$0^2 \neq 2 + x \cdot 0$ for any $x$. | $2 : \begin{cases} 1 : y = 0 \\ 1 : \text{explanation} \end{cases}$ |
| (d) When $y = 3$, $3^2 = 2 + 3x$ so $x = \dfrac{7}{3}$.<br>$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{y}{2y - x} \cdot \dfrac{dx}{dt}$<br>At $t = 5$, $6 = \dfrac{3}{6 - \frac{7}{3}} \cdot \dfrac{dx}{dt} = \dfrac{9}{11} \cdot \dfrac{dx}{dt}$<br>$\left.\dfrac{dx}{dt}\right\|_{t=5} = \dfrac{22}{3}$ | $3 : \begin{cases} 1 : \text{solves for } x \\ 1 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$ |

## Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point $P$ with $x$-coordinate 3 at which the line tangent to the curve at $P$ is horizontal. Find the $y$-coordinate of $P$.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point $P$ found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point $P$? Justify your answer.

| | |
|---|---|
| (a) $2x + 8yy' = 3y + 3xy'$<br>$(8y - 3x)y' = 3y - 2x$<br>$y' = \frac{3y - 2x}{8y - 3x}$ | $2: \begin{cases} 1: \text{implicit differentiation} \\ 1: \text{solves for } y' \end{cases}$ |
| (b) $\frac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$<br>When $x = 3,\ 3y = 6$<br>$y = 2$<br>$3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$<br>Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point. | $3: \begin{cases} 1: \frac{dy}{dx} = 0 \\ 1: \text{shows slope is 0 at } (3, 2) \\ 1: \text{shows } (3, 2) \text{ lies on curve} \end{cases}$ |
| (c) $\frac{d^2y}{dx^2} = \frac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$<br>At $P = (3, 2)$, $\frac{d^2y}{dx^2} = \frac{(16 - 9)(-2)}{(16 - 9)^2} = -\frac{2}{7}$.<br>Since $y' = 0$ and $y'' < 0$ at $P$, the curve has a local maximum at $P$. | $4: \begin{cases} 2: \frac{d^2y}{dx^2} \\ 1: \text{value of } \frac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1: \text{conclusion with justification} \end{cases}$ |







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## Relationships between f' and f

### Absolute Extrema & Critical Numbers

13 44. What is the minimum value of $f(x) = x \ln x$?

(A) $-e$ (B) $-1$ (C) $-\frac{1}{e}$ (D) $0$ (E) $f(x)$ has no minimum value.

88 33. The absolute maximum value of $f(x) = x^3 - 3x^2 + 12$ on the closed interval $[-2, 4]$ occurs at $x =$

(A) 4 (B) 2 (C) 1 (D) 0 (E) $-2$

73 BC 27. If $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 5$ and the domain is the set of all $x$ such that $0 \le x \le 9$, then the absolute maximum value of the function $f$ occurs when $x$ is

(A) 0 (B) 2 (C) 4 (D) 6 (E) 9

73 29. Let $f(x) = \left| \sin x - \frac{1}{2} \right|$. The maximum value attained by $f$ is

(A) $\frac{1}{2}$ (B) $1$ (C) $\frac{3}{2}$ (D) $\frac{\pi}{2}$ (E) $\frac{3\pi}{2}$

85 36. If $f$ is a continuous function defined for all real numbers $x$ and if the maximum value of $f(x)$ is 5 and the minimum value of $f(x)$ is $-7$, then which of the following must be true?

I. The maximum value of $f(|x|)$ is 5.
II. The maximum value of $|f(x)|$ is 7.
III. The minimum value of $f(|x|)$ is 0.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

93 23. How many critical points does the function $f(x) = (x+2)^5(x-3)^4$ have?

(A) One (B) Two (C) Three (D) Five (E) Nine

88 37. If $f(x) = e^x \sin x$, then the number of zeros of $f$ on the closed interval $[0, 2\pi]$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

# Increasing / Decreasing Intervals

98 BC 1. What are all values of $x$ for which the function $f$ defined by $f(x) = x^3 + 3x^2 - 9x + 7$ is increasing?

(A) $-3 < x < 1$
(B) $-1 < x < 1$
(C) $x < -3$ or $x > 1$
(D) $x < -1$ or $x > 3$
(E) All real numbers

73 BC 3. If $f(x) = x + \frac{1}{x}$, then the set of values for which $f$ increases is

(A) $(-\infty, -1] \cup [1, \infty)$
(B) $[-1, 1]$
(C) $(-\infty, \infty)$
(D) $(0, \infty)$
(E) $(-\infty, 0) \cup (0, \infty)$

97 13. Let $f$ be a function defined for all real numbers $x$. If $f'(x) = \frac{|4 - x^2|}{x - 2}$, then $f$ is decreasing on the interval

(A) $(-\infty, 2)$
(B) $(-\infty, \infty)$
(C) $(-2, 4)$
(D) $(-2, \infty)$
(E) $(2, \infty)$

69 AB 21. At $x = 0$, which of the following is true of the function $f$ defined by $f(x) = x^2 + e^{-2x}$?

(A) $f$ is increasing.
(B) $f$ is decreasing.
(C) $f$ is discontinuous.
(D) $f$ has a relative minimum.
(E) $f$ has a relative maximum.

97 22. What are all values of $x$ for which the function $f$ defined by $f(x) = (x^2 - 3)e^{-x}$ is increasing?

(A) There are no such values of $x$.
(B) $x < -1$ and $x > 3$
(C) $-3 < x < 1$
(D) $-1 < x < 3$
(E) All values of $x$

93 27. The function $f$ given by $f(x) = x^3 + 12x - 24$ is

(A) increasing for $x < -2$, decreasing for $-2 < x < 2$, increasing for $x > 2$
(B) decreasing for $x < 0$, increasing for $x > 0$
(C) increasing for all $x$
(D) decreasing for all $x$
(E) decreasing for $x < -2$, increasing for $-2 < x < 2$, decreasing for $x > 2$

93 BC 22. If $f(x) = x^2 e^x$, then the graph of $f$ is decreasing for all $x$ such that

(A) $x < -2$ (B) $-2 < x < 0$ (C) $x > -2$ (D) $x < 0$ (E) $x > 0$

85 39. If $f(x) = \frac{\ln x}{x}$, for all $x > 0$, which of the following is true?

(A) $f$ is increasing for all $x$ greater than 0.
(B) $f$ is increasing for all $x$ greater than 1.
(C) $f$ is decreasing for all $x$ between 0 and 1.
(D) $f$ is decreasing for all $x$ between 1 and $e$.
(E) $f$ is decreasing for all $x$ greater than $e$.

98 22. The function $f$ is given by $f(x) = x^4 + x^2 - 2$. On which of the following intervals is $f$ increasing?

(A) $\left(-\frac{1}{\sqrt{2}}, \infty\right)$

(B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(C) $(0, \infty)$

(D) $(-\infty, 0)$

(E) $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$

# Relative Extrema

98 BC 23. Let $f$ be a function defined and continuous on the closed interval $[a,b]$. If $f$ has a relative maximum at $c$ and $a<c<b$, which of the following statements must be true?

I. $f'(c)$ exists.
II. If $f'(c)$ exists, then $f'(c)=0$.
III. If $f''(c)$ exists, then $f''(c)\le 0$.

(A) II only (B) III only (C) I and II only (D) I and III only (E) II and III only

69 AB 7. For what value of $k$ will $x+\frac{k}{x}$ have a relative maximum at $x=-2$?

(A) −4 (B) −2 (C) 2 (D) 4 (E) None of these

69 AB 30. If a function $f$ is continuous for all $x$ and if $f$ has a relative maximum at $(-1,4)$ and a relative minimum at $(3,-2)$, which of the following statements must be true?

(A) The graph of $f$ has a point of inflection somewhere between $x=-1$ and $x=3$.
(B) $f'(-1)=0$
(C) The graph of $f$ has a horizontal asymptote.
(D) The graph of $f$ has a horizontal tangent line at $x=3$.
(E) The graph of $f$ intersects both axes.

85 16. The function defined by $f(x)=x^3-3x^2$ for all real numbers $x$ has a relative maximum at $x=$

(A) −2 (B) 0 (C) 1 (D) 2 (E) 4

97 BC 3. The function $f$ given by $f(x)=3x^5-4x^3-3x$ has a relative maximum at $x=$

(A) −1 (B) $-\frac{\sqrt{5}}{5}$ (C) 0 (D) $\frac{\sqrt{5}}{5}$ (E) 1

93 15. For what value of $x$ does the function $f(x)=(x-2)(x-3)^2$ have a relative maximum?

(A) −3 (B) $-\frac{7}{3}$ (C) $-\frac{5}{2}$ (D) $\frac{7}{3}$ (E) $\frac{5}{2}$

69 BC 7. For what value of $k$ will $x+\frac{k}{x}$ have a relative maximum at $x=-2$?

(A) $-4$ (B) $-2$ (C) 2 (D) 4 (E) None of these

69 BC 21. At $x=0$, which of the following is true of the function $f$ defined by $f(x)=x^2+e^{-2x}$ ?

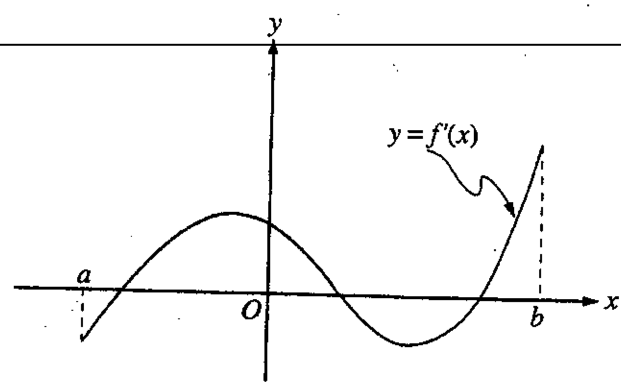
(A) $f$ is increasing.
(B) $f$ is decreasing.
(C) $f$ is discontinuous.
(D) $f$ has a relative minimum.
(E) $f$ has a relative maximum.

88 BC 19. A polynomial $p(x)$ has a relative maximum at $(-2,4)$, a relative minimum at $(1,1)$, a relative maximum at $(5,7)$ and no other critical points. How many zeros does $p(x)$ have?

(A) One (B) Two (C) Three (D) Four (E) Five

85 BC 2. At what values of $x$ does $f(x)=3x^5-5x^3+15$ have a relative maximum?

(A) $-1$ only (B) 0 only (C) 1 only (D) $-1$ and 1 only (E) $-1$, 0 and 1




97 BC 12. The graph of $f'$, the derivative of $f$, is shown in the figure above. Which of the following describes all relative extrema of $f$ on the open interval $(a,b)$?

(A) One relative maximum and two relative minima
(B) Two relative maxima and one relative minimum
(C) Three relative maxima and one relative minimum
(D) One relative maximum and three relative minima
(E) Three relative maxima and two relative minima

93 BC 9. If $f(x) = 1 + x^{\frac{2}{3}}$, which of the following is NOT true?

(A) $f$ is continuous for all real numbers.

(B) $f$ has a minimum at $x = 0$.

(C) $f$ is increasing for $x > 0$.

(D) $f'(x)$ exists for all $x$.

(E) $f''(x)$ is negative for $x > 0$.

93 BC 14. The derivative of $f$ is $x^4(x-2)(x+3)$. At how many points will the graph of $f$ have a relative maximum?

(A) None (B) One (C) Two (D) Three (E) Four

88 20. Let $f$ be a polynomial function with degree greater than 2. If $a \neq b$ and $f(a) = f(b) = 1$, which of the following must be true for at least one value of $x$ between $a$ and $b$?

I. $f(x) = 0$
II. $f'(x) = 0$
III. $f''(x) = 0$

(A) None (B) I only (C) II only (D) I and II only (E) I, II, and III

# Concavity and Inflection

98 1. What is the $x$-coordinate of the point of inflection on the graph of $y = \frac{1}{3}x^3 + 5x^2 + 24$?

(A) 5 (B) 0 (C) $-\frac{10}{3}$ (D) $-5$ (E) $-10$

69 BC 2. What are the coordinates of the inflection point on the graph of $y = (x+1)\arctan x$?

(A) $(-1,0)$ (B) $(0,0)$ (C) $(0,1)$ (D) $\left(1,\frac{\pi}{4}\right)$ (E) $\left(1,\frac{\pi}{2}\right)$

98 BC 16. If $f$ is the function defined by $f(x) = 3x^5 - 5x^4$, what are all the $x$-coordinates of points of inflection for the graph of $f$?

(A) $-1$ (B) 0 (C) 1 (D) 0 and 1 (E) $-1$, 0, and 1

69 AB 17. The graph of $y = 5x^4 - x^5$ has a point of inflection at

(A) $(0,0)$ only (B) $(3,162)$ only (C) $(4,256)$ only (D) $(0,0)$ and $(3,162)$ (E) $(0,0)$ and $(4,256)$

98 19. If $f''(x) = x(x+1)(x-2)^2$, then the graph of $f$ has inflection points when $x =$

(A) $-1$ only (B) 2 only (C) $-1$ and 0 only (D) $-1$ and 2 only (E) $-1$, 0, and 2 only

97 5. The graph of $y = 3x^4 - 16x^3 + 24x^2 + 48$ is concave down for

(A) $x < 0$

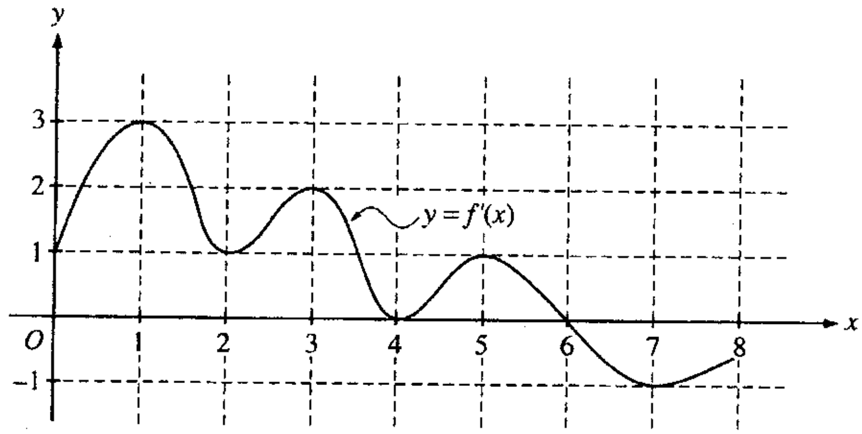
(B) $x > 0$

(C) $x < -2$ or $x > -\frac{2}{3}$

(D) $x < \frac{2}{3}$ or $x > 2$

(E) $\frac{2}{3} < x < 2$

Questions 7-9 refer to the graph and the information below.




The function $f$ is defined on the closed interval $[0,8]$. The graph of its derivative $f'$ is shown above.

97 BC 7. The point $(3,5)$ is on the graph of $y = f(x)$. An equation of the line tangent to the graph of $f$ at $(3,5)$ is

(A) $y = 2$

(B) $y = 5$

(C) $y - 5 = 2(x - 3)$

(D) $y + 5 = 2(x - 3)$

(E) $y + 5 = 2(x + 3)$

97 BC 8. ~~How many points of inflection does the graph of $f$ have?~~

(A) Two
(B) Three
(C) Four
(D) Five
(E) Six

97 BC 9. At what value of $x$ does the absolute minimum of $f$ occur?

(A) 0
(B) 2
(C) 4
(D) 6
(E) 8

73 BC 26. Which of the following is true about the graph of $y = \ln|x^2 - 1|$ in the interval $(-1,1)$?

(A) It is increasing.
(B) It attains a relative minimum at $(0,0)$.
(C) It has a range of all real numbers.
(D) It is concave down.
(E) It has an asymptote of $x = 0$.

73 22. Given the function defined by $f(x) = 3x^5 - 20x^3$, find all values of $x$ for which the graph of $f$ is concave up.

(A) $x > 0$
(B) $-\sqrt{2} < x < 0$ or $x > \sqrt{2}$
(C) $-2 < x < 0$ or $x > 2$
(D) $x > \sqrt{2}$
(E) $-2 < x < 2$

69 BC 17. The graph of $y = 5x^4 - x^5$ has a point of inflection at

(A) $(0,0)$ only (B) $(3,162)$ only (C) $(4,256)$ only
(D) $(0,0)$ and $(3,162)$ (E) $(0,0)$ and $(4,256)$

88 BC 27. If the graph of $y = x^3 + ax^2 + bx - 4$ has a point of inflection at $(1,-6)$, what is the value of $b$?

(A) $-3$ (B) $0$ (C) $1$ (D) $3$
(E) It cannot be determined from the information given.

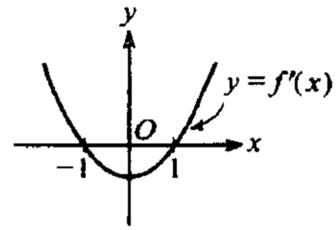
88 4. The graph of $y = \frac{-5}{x-2}$ is concave downward for all values of $x$ such that

(A) $x < 0$ (B) $x < 2$ (C) $x < 5$ (D) $x > 0$ (E) $x > 2$

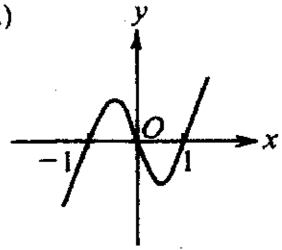
93 21. At what value of $x$ does the graph of $y = \frac{1}{x^2} - \frac{1}{x^3}$ have a point of inflection?

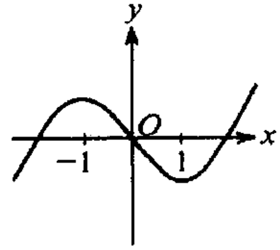
(A) 0 (B) 1 (C) 2 (D) 3 (E) At no value of $x$

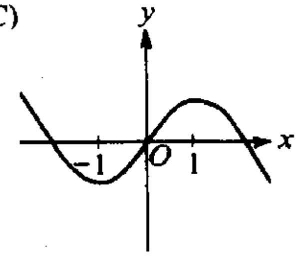
Graphical relationship between f, f', and f''

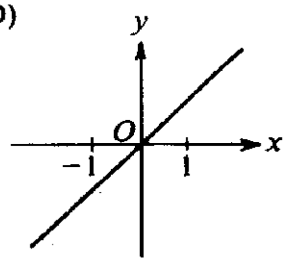


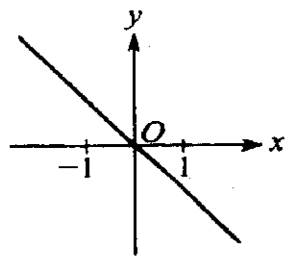

85 33. The graph of the derivative of $f$ is shown in the figure above. Which of the following could be the graph of $f$?

(A)


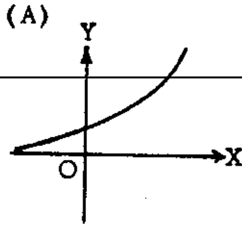
(B)


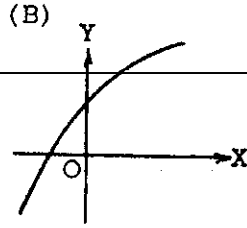
(C)


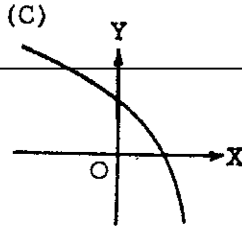
(D)


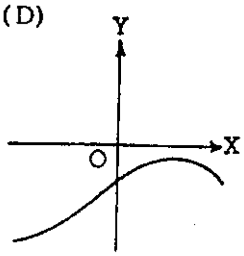
(E)


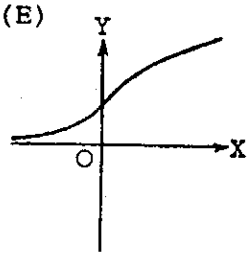
69 BC 16. If $y$ is a function $x$ such that $y' > 0$ for all $x$ and $y'' < 0$ for all $x$, which of the following could be part of the graph of $y = f(x)$?

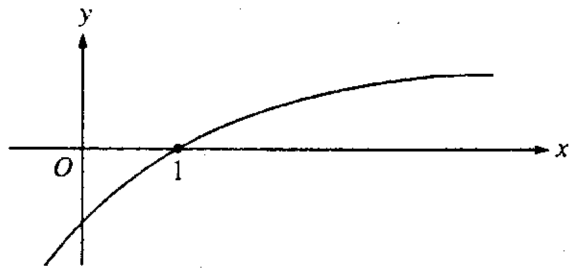
(A)


(B)


(C)


(D)


(E)





98 BC 17. The graph of a twice-differentiable function $f$ is shown in the figure above. Which of the following is true?

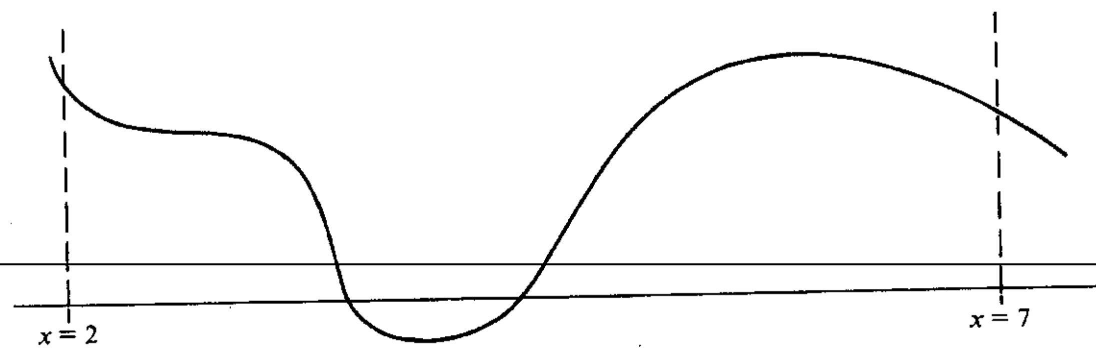
(A) $f(1) < f'(1) < f''(1)$

(B) $f(1) < f''(1) < f'(1)$

(C) $f'(1) < f(1) < f''(1)$

(D) $f''(1) < f(1) < f'(1)$

(E) $f''(1) < f'(1) < f(1)$



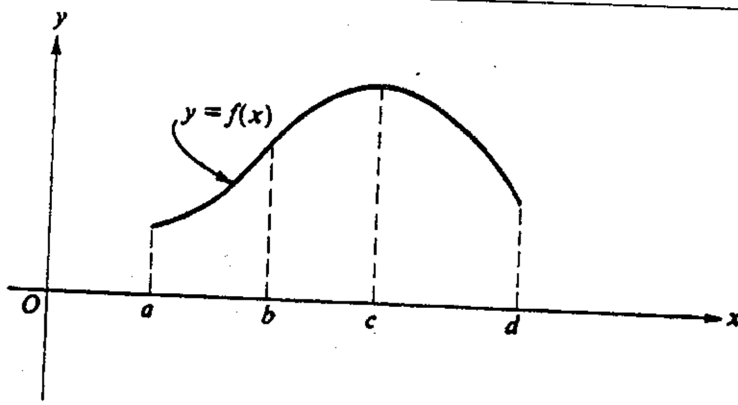

85 BC 20. The graph of $y = f(x)$ on the closed interval $[2,7]$ is shown above. How many points of inflection does this graph have on this interval?

(A) One (B) Two (C) Three (D) Four (E) Five

98 81. Let $f$ be the function given by $f(x) = |x|$. Which of the following statements about $f$ are true?

I. $f$ is continuous at $x = 0$.
II. $f$ is differentiable at $x = 0$.
III. $f$ has an absolute minimum at $x = 0$.

(A) I only (B) II only (C) III only (D) I and III only (E) II and III only



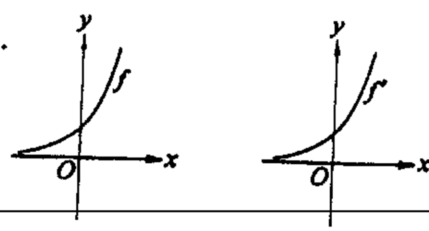

88 8. The graph of $y = f(x)$ is shown in the figure above. On which of the following intervals are $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$?

I. $a < x < b$
II. $b < x < c$
III. $c < x < d$

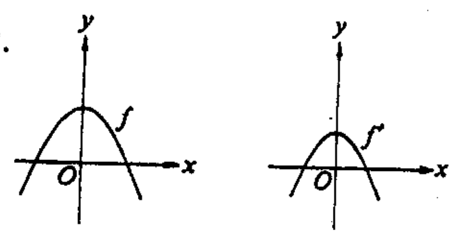
(A) I only (B) II only (C) III only (D) I and II (E) II and III

88 BC 9. Which of the following pairs of graphs could represent the graph of a function and the graph of its derivative?

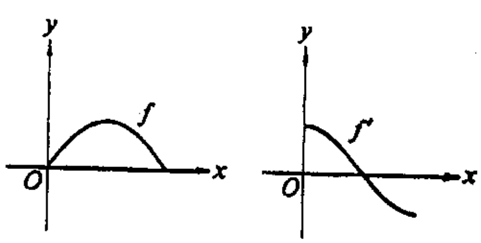
I.




II.

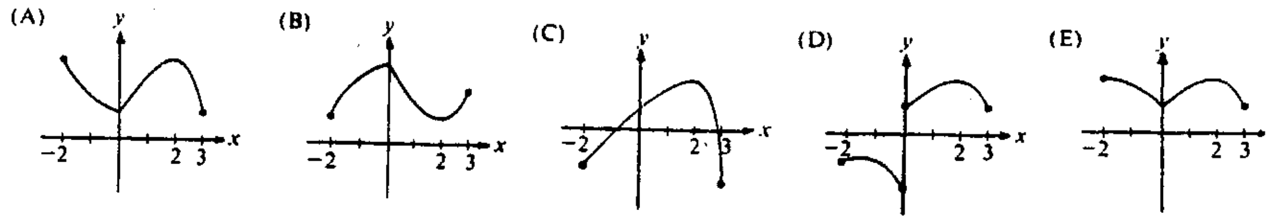


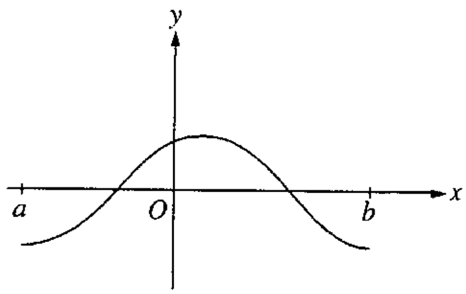

III.




(A) I only (B) II only (C) III only (D) I and III (E) II and III

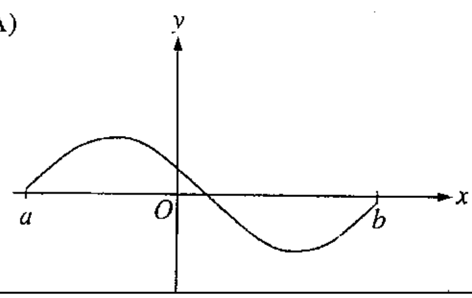
85 BC 43. Let $f$ be a function that is continuous on the closed interval $[-2,3]$ such that $f'(0)$ does not exist, $f'(2)=0$, and $f''(x)<0$ for all $x$ except $x=0$. Which of the following could be the graph of $f$?



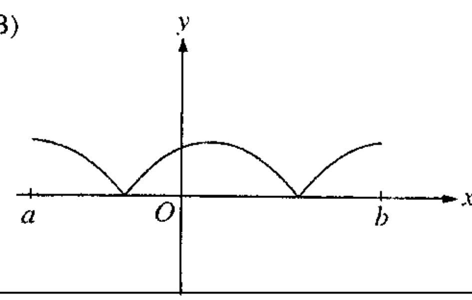




98 23. The graph of $f$ is shown in the figure above. Which of the following could be the graph of the derivative of $f$?

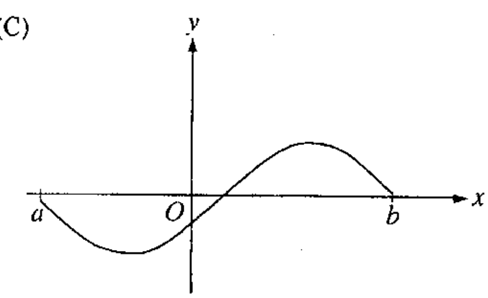
(A)



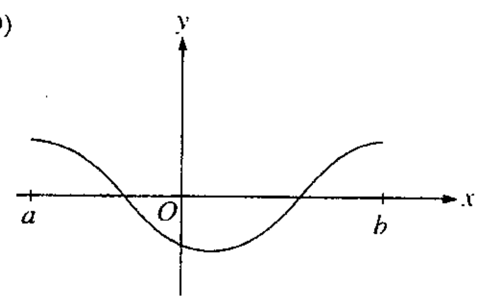
(B)



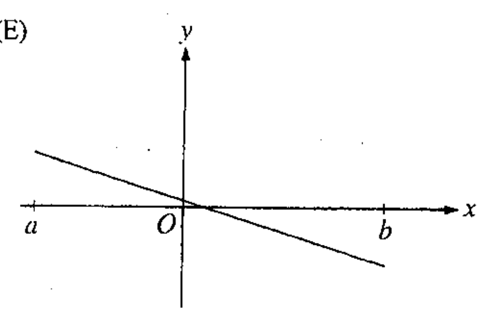
(C)

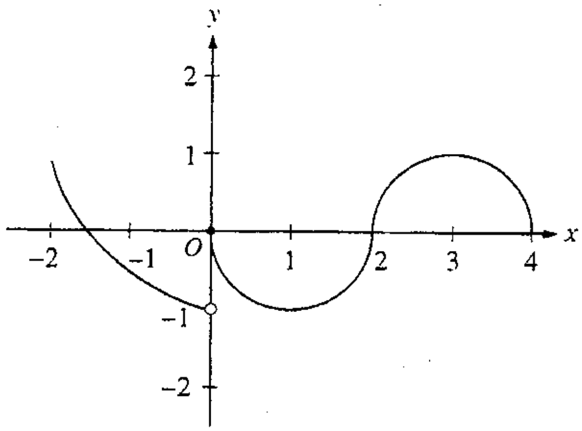


(D)



(E)



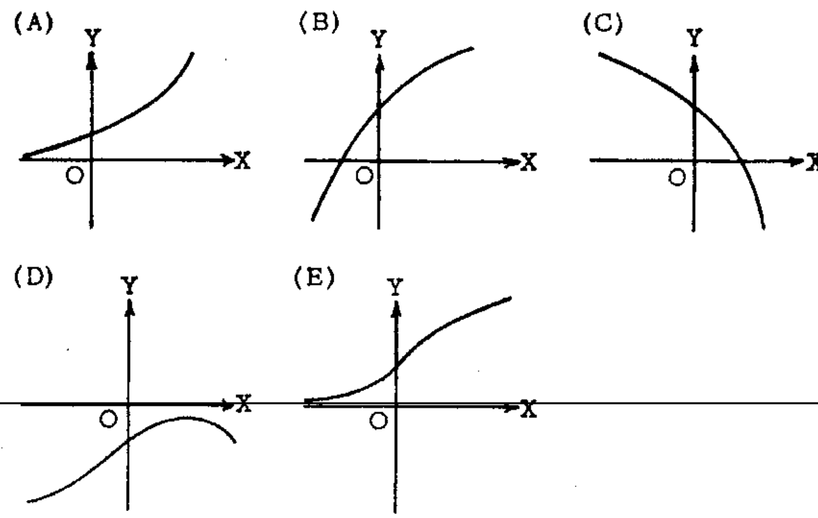



13. The graph of the function $f$ shown in the figure above has a vertical tangent at the point $(2,0)$ and horizontal tangents at the points $(1,-1)$ and $(3,1)$. For what values of $x$, $-2 < x < 4$, is $f$ not differentiable?

(A) 0 only (B) 0 and 2 only (C) 1 and 3 only (D) 0, 1, and 3 only (E) 0, 1, 2, and 3

---

69 AB 16. If $y$ is a function of $x$ such that $y' > 0$ for all $x$ and $y'' < 0$ for all $x$, which of the following could be part of the graph of $y = f(x)$?

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93 19. Let $f$ be the function defined by $f(x) = \begin{cases} x^3 & \text{for } x \le 0, \\ x & \text{for } x > 0. \end{cases}$ Which of the following statements about $f$ is true?

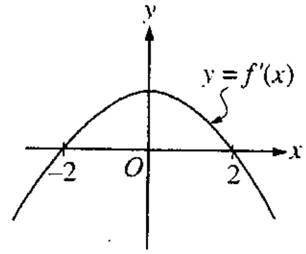
(A) $f$ is an odd function.

(B) $f$ is discontinuous at $x = 0$.

(C) $f$ has a relative maximum.

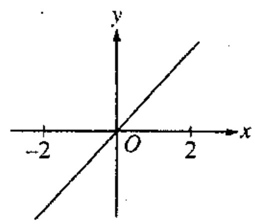
(D) $f'(0) = 0$

(E) $f'(x) > 0$ for $x \ne 0$

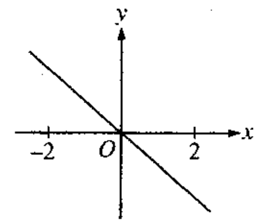



97 11. The graph of the derivative of $f$ is shown in the figure above. Which of the following could be the graph of $f$?

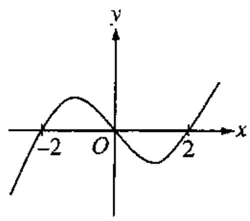
(A)



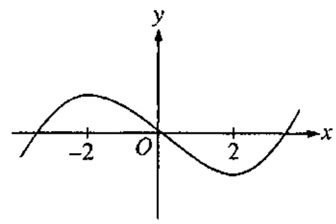

(B)



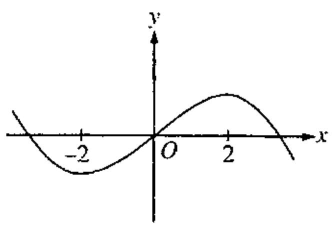

(C)

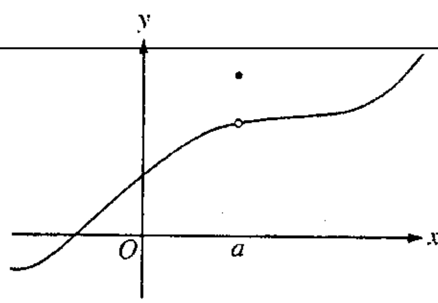



(D)




(E)




98 76. The graph of a function $f$ is shown above. Which of the following statements about $f$ is false?

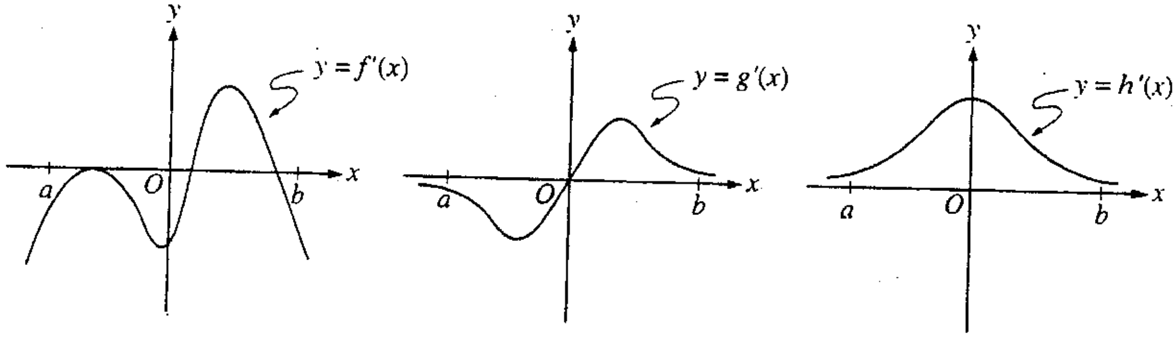
(A) $f$ is continuous at $x = a$.

(B) $f$ has a relative maximum at $x = a$.

(C) $x = a$ is in the domain of $f$.

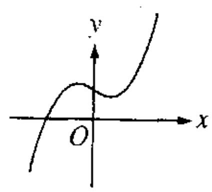
(D) $\lim_{x \to a^+} f(x)$ is equal to $\lim_{x \to a^-} f(x)$.

(E) $\lim_{x \to a} f(x)$ exists.



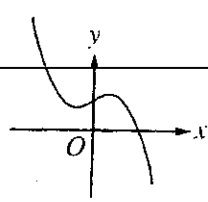

98 79. The graphs of the derivatives of the functions $f$, $g$, and $h$ are shown above. Which of the functions $f$, $g$, or $h$ have a relative maximum on the open interval $a < x < b$?

(A) $f$ only
(B) $g$ only
(C) $h$ only
(D) $f$ and $g$ only
(E) $f$, $g$, and $h$

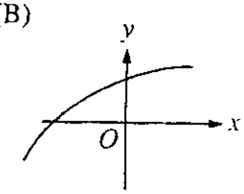



98 BC 6. The graph of $y = h(x)$ is shown above. Which of the following could be the graph of $y = h'(x)$?

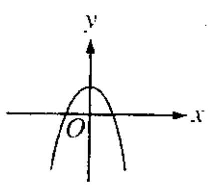
(A)



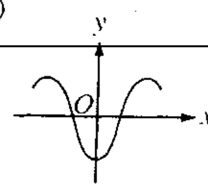

(B)



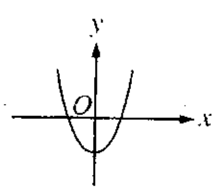

(C)




(D)




(E)



97 77. The graph of the function $y = x^3 + 6x^2 + 7x - 2\cos x$ changes concavity at $x =$

(A) $-1.58$ (B) $-1.63$ (C) $-1.67$ (D) $-1.89$ (E) $-2.33$

97 BC 80. Let $f$ be the function given by $f(x) = \cos(2x) + \ln(3x)$. What is the least value of $x$ at which the graph of $f$ changes concavity?

(A) 0.56 (B) 0.93 (C) 1.18 (D) 2.38 (E) 2.44

98 80. The first derivative of the function $f$ is given by $f'(x) = \frac{\cos^2 x}{x} - \frac{1}{5}$. How many critical values does $f$ have on the open interval $(0,10)$?

(A) One
(B) Three
(C) Four
(D) Five
(E) Seven

97 85. If the derivative of $f$ is given by $f'(x) = e^x - 3x^2$, at which of the following values of $x$ does $f$ have a relative maximum value?

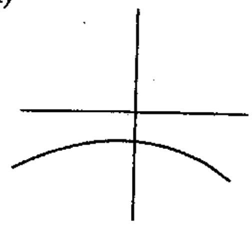
(A) $-0.46$ (B) 0.20 (C) 0.91 (D) 0.95 (E) 3.73

98 89. If $g$ is a differentiable function such that $g(x) < 0$ for all real numbers $x$ and if $f'(x) = (x^2 - 4)g(x)$, which of the following is true?

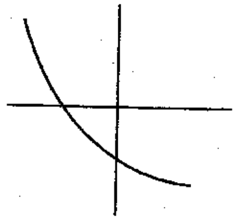
(A) $f$ has a relative maximum at $x = -2$ and a relative minimum at $x = 2$.
(B) $f$ has a relative minimum at $x = -2$ and a relative maximum at $x = 2$.
(C) $f$ has relative minima at $x = -2$ and at $x = 2$.
(D) $f$ has relative maxima at $x = -2$ and at $x = 2$.
(E) It cannot be determined if $f$ has any relative extrema.

16. For which curve shown below are both $f'$ and $f''$ negative?

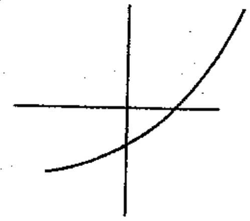
(A)



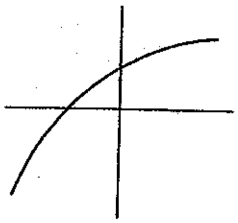
(B)



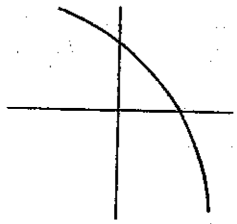
(C)



(D)

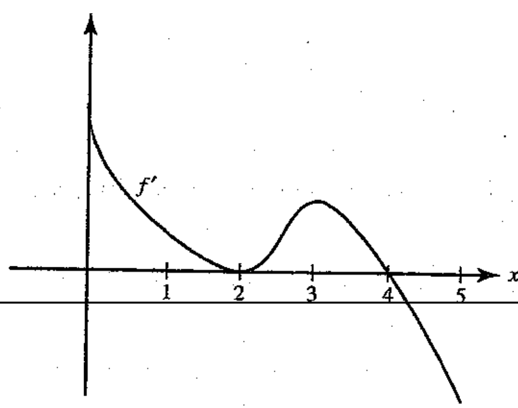


(E)



17. For which curve shown in question 16 is $f''$ positive but $f'$ negative?

Questions 31 and 32. Refer to the graph of $f'$ below.




31. $f$ has a local minimum at $x =$

**(A)** 0 only **(B)** 4 only **(C)** 0 and 4 **(D)** 0 and 5 **(E)** 0, 4, and 5

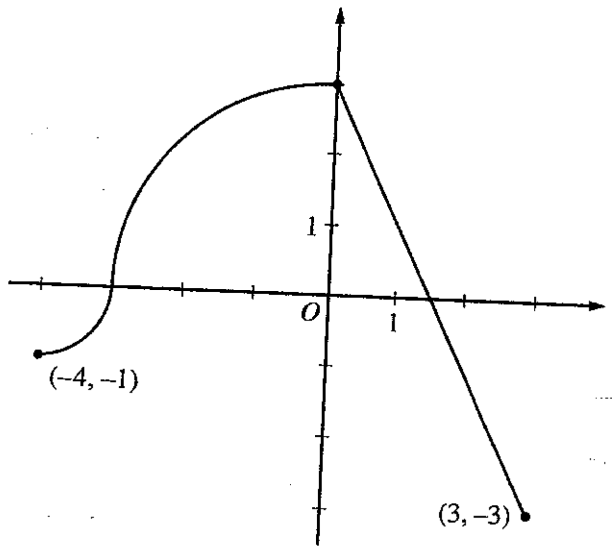
32. The graph of $f$ has a point of inflection at $x =$

**(A)** 2 only **(B)** 3 only **(C)** 4 only
**(D)** 2 and 3 only **(E)** 2, 3, and 4


57

# Relationships between f, f', and f'' FRQs

(Mostly) w/o 2 FTC




11-A

Graph of $f$

4. The continuous function $f$ is defined on the interval $-4 \le x \le 3$. The graph of $f$ consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the $x$-coordinate of the point at which $g$ has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of $x$ on the interval $-4 < x < 3$ for which the graph of $g$ has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of $f$ on the interval $-4 \le x \le 3$. There is no point $c$, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

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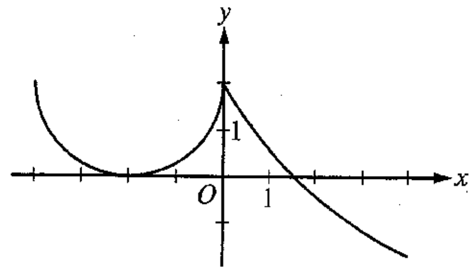
11-B

4. Consider a differentiable function $f$ having domain all positive real numbers, and for which it is known that $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

(a) Find the $x$-coordinate of the critical point of $f$. Determine whether the point is a relative maximum, a relative minimum, or neither for the function $f$. Justify your answer.

(b) Find all intervals on which the graph of $f$ is concave down. Justify your answer.

(c) Given that $f(1) = 2$, determine the function $f$.




Graph of $f'$

9- A

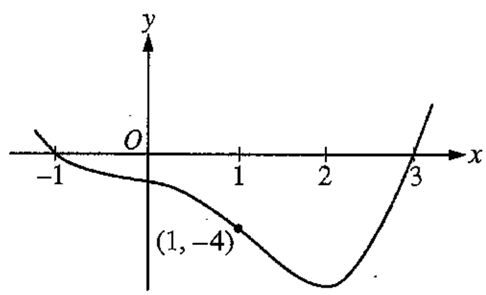
6. The derivative of a function $f$ is defined by $f'(x) = \begin{cases} g(x) & \text{for } -4 \le x \le 0 \\ 5e^{-x/3} - 3 & \text{for } 0 < x \le 4 \end{cases}$.

   The graph of the continuous function $f'$, shown in the figure above, has $x$-intercepts at $x = -2$ and $x = 3\ln\left(\frac{5}{3}\right)$. The graph of $g$ on $-4 \le x \le 0$ is a semicircle, and $f(0) = 5$.

   (a) For $-4 < x < 4$, find all values of $x$ at which the graph of $f$ has a point of inflection. Justify your answer.

   (b) Find $f(-4)$ and $f(4)$.

   (c) For $-4 \le x \le 4$, find the value of $x$ at which $f$ has an absolute maximum. Justify your answer.




Graph of $f'$

9-B

5. Let $f$ be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of $f'$, the derivative of $f$, is shown above. The graph of $f'$ crosses the $x$-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let $g$ be the function given by $g(x) = e^{f(x)}$.

   (a) Write an equation for the line tangent to the graph of $g$ at $x = 1$.

   (b) For $-1.2 < x < 3.2$, find all values of $x$ at which $g$ has a local maximum. Justify your answer.

   (c) The second derivative of $g$ is $g''(x) = e^{f(x)}\left[\left(f'(x)\right)^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

   (d) Find the average rate of change of $g'$, the derivative of $g$, over the interval $[1, 3]$.

8-A

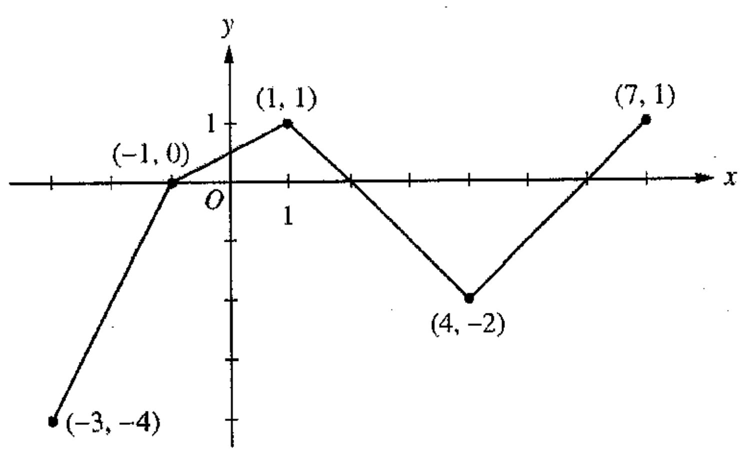
6. Let $f$ be the function given by $f(x) = \frac{\ln x}{x}$ for all $x > 0$. The derivative of $f$ is given by $f'(x) = \frac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of $f$ at $x = e^2$.

(b) Find the $x$-coordinate of the critical point of $f$. Determine whether this point is a relative minimum, a relative maximum, or neither for the function $f$. Justify your answer.

(c) The graph of the function $f$ has exactly one point of inflection. Find the $x$-coordinate of this point.

(d) Find $\lim_{x \to 0^+} f(x)$.




Graph of $g'$

8-B

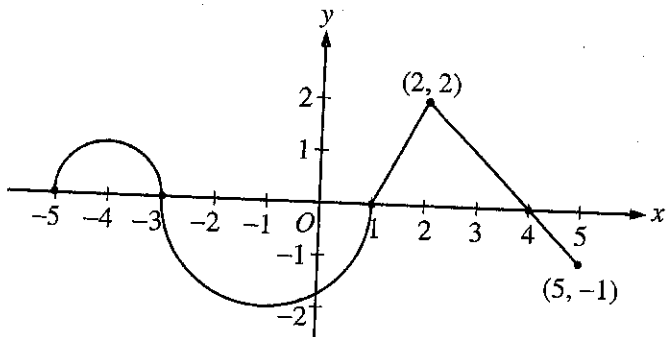
5. Let $g$ be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function $g'$, the derivative of $g$, is shown above for $-3 \le x \le 7$.

(a) Find the $x$-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of $g$ on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of $c$, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?




7-B

Graph of $f'$

4. Let $f$ be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of $f'$, the derivative of $f$, consists of two semicircles and two line segments, as shown above.

(a) For $-5 < x < 5$, find all values $x$ at which $f$ has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values $x$ at which the graph of $f$ has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of $f$ is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

| $x$ | 0 | $0 < x < 1$ | 1 | $1 < x < 2$ | 2 | $2 < x < 3$ | 3 | $3 < x < 4$ |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | −1 | Negative | 0 | Positive | 2 | Positive | 0 | Negative |
| $f'(x)$ | 4 | Positive | 0 | Positive | DNE | Negative | −3 | Negative |
| $f''(x)$ | −2 | Negative | 0 | Positive | DNE | Negative | 0 | Positive |

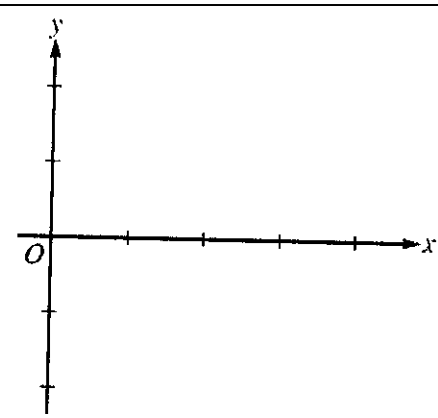
5-A

4. Let $f$ be a function that is continuous on the interval $[0, 4)$. The function $f$ is twice differentiable except at $x = 2$. The function $f$ and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of $f$ do not exist at $x = 2$.

(a) For $0 < x < 4$, find all values of $x$ at which $f$ has a relative extremum. Determine whether $f$ has a relative maximum or a relative minimum at each of these values. Justify your answer.

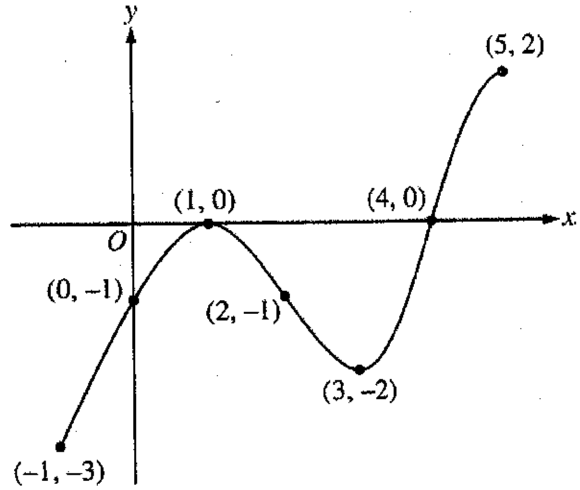
(b) On the axes provided, sketch the graph of a function that has all the characteristics of $f$.

**(Note: Use the axes provided in the pink test booklet.)**




(c) Let $g$ be the function defined by $g(x) = \int_1^x f(t)\,dt$ on the open interval $(0, 4)$. For $0 < x < 4$, find all values of $x$ at which $g$ has a relative extremum. Determine whether $g$ has a relative maximum or a relative minimum at each of these values. Justify your answer.

(d) For the function $g$ defined in part (c), find all values of $x$, for $0 < x < 4$, at which the graph of $g$ has a point of inflection. Justify your answer.



Graph of $f'$

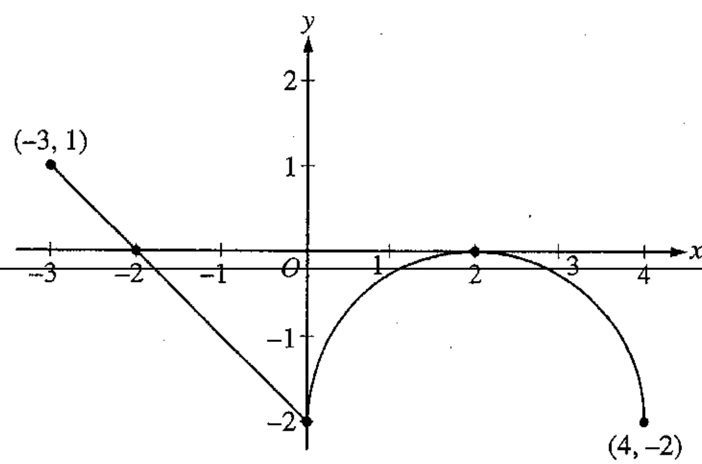
4-B

4. The figure above shows the graph of $f'$, the derivative of the function $f$, on the closed interval $-1 \le x \le 5$. The graph of $f'$ has horizontal tangent lines at $x = 1$ and $x = 3$. The function $f$ is twice differentiable with $f(2) = 6$.

   (a) Find the $x$-coordinate of each of the points of inflection of the graph of $f$. Give a reason for your answer.

   (b) At what value of $x$ does $f$ attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of $x$ does $f$ attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

   (c) Let $g$ be the function defined by $g(x) = x f(x)$. Find an equation for the line tangent to the graph of $g$ at $x = 2$.



Graph of $f'$

3-A

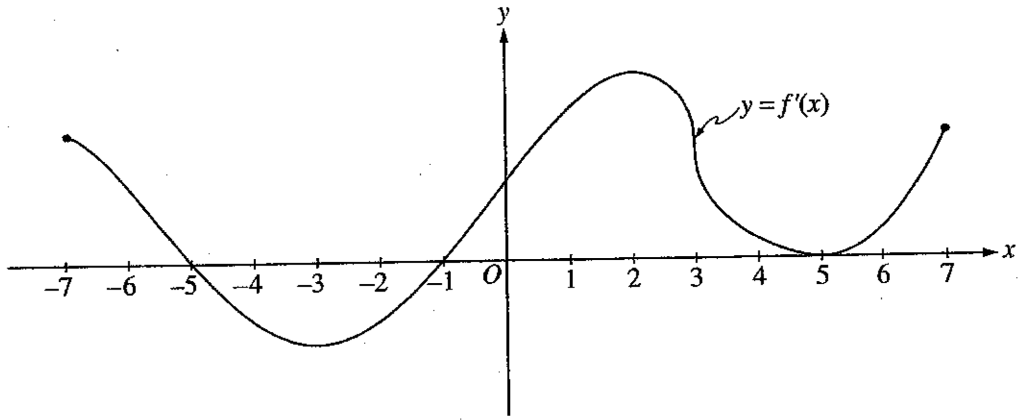
4. Let $f$ be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of $f'$, the derivative of $f$, consists of one line segment and a semicircle, as shown above.

   (a) On what intervals, if any, is $f$ increasing? Justify your answer.

   (b) Find the $x$-coordinate of each point of inflection of the graph of $f$ on the open interval $-3 < x < 4$. Justify your answer.

   (c) Find an equation for the line tangent to the graph of $f$ at the point $(0, 3)$.

   (d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

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3. The figure above shows the graph of $f'$, the derivative of the function $f$, for $-7 \le x \le 7$. The graph of $f'$ has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

   (a) Find all values of $x$, for $-7 < x < 7$, at which $f$ attains a relative minimum. Justify your answer.

   (b) Find all values of $x$, for $-7 < x < 7$, at which $f$ attains a relative maximum. Justify your answer.

   (c) Find all values of $x$, for $-7 < x < 7$, at which $f''(x) < 0$.

   (d) At what value of $x$, for $-7 \le x \le 7$, does $f$ attain its absolute maximum? Justify your answer.

7-A 6. Let $f$ be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where $k$ is a positive constant.

   (a) Find $f'(x)$ and $f''(x)$.

   (b) For what value of the constant $k$ does $f$ have a critical point at $x = 1$? For this value of $k$, determine whether $f$ has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

   (c) For a certain value of the constant $k$, the graph of $f$ has a point of inflection on the $x$-axis. Find this value of $k$.

# Relationships between f, f', and f''

## WITH CALCULATOR

10-B

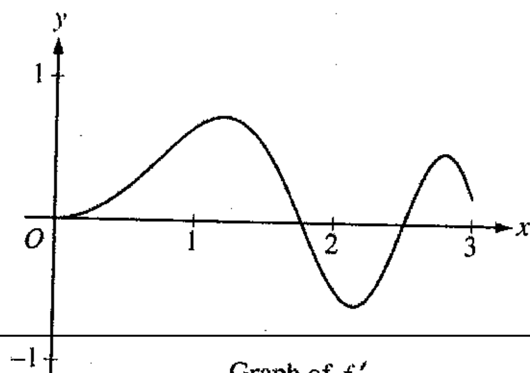
2. The function $g$ is defined for $x > 0$ with $g(1) = 2$, $g'(x) = \sin\left(x + \frac{1}{x}\right)$, and $g''(x) = \left(1 - \frac{1}{x^2}\right)\cos\left(x + \frac{1}{x}\right)$.

   (a) Find all values of $x$ in the interval $0.12 \le x \le 1$ at which the graph of $g$ has a horizontal tangent line.

   (b) On what subintervals of $(0.12, 1)$, if any, is the graph of $g$ concave down? Justify your answer.

   (c) Write an equation for the line tangent to the graph of $g$ at $x = 0.3$.

   (d) Does the line tangent to the graph of $g$ at $x = 0.3$ lie above or below the graph of $g$ for $0.3 < x < 1$? Why?




6-B

Graph of $f'$

2. Let $f$ be the function defined for $x \ge 0$ with $f(0) = 5$ and $f'$, the first derivative of $f$, given by $f'(x) = e^{(-x/4)}\sin\left(x^2\right)$. The graph of $y = f'(x)$ is shown above.

   (a) Use the graph of $f'$ to determine whether the graph of $f$ is concave up, concave down, or neither on the interval $1.7 < x < 1.9$. Explain your reasoning.

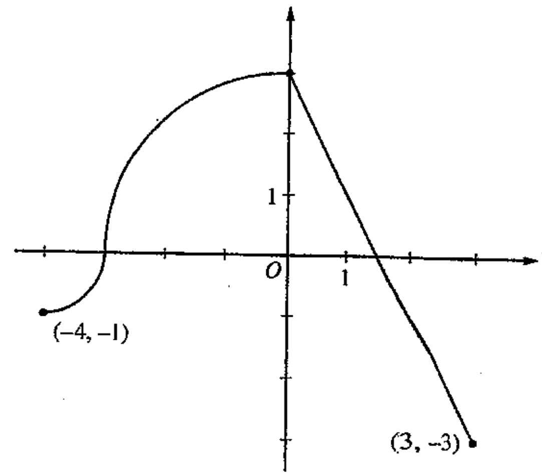
   (b) On the interval $0 \le x \le 3$, find the value of $x$ at which $f$ has an absolute maximum. Justify your answer.

   (c) Write an equation for the line tangent to the graph of $f$ at $x = 2$.

# AP® CALCULUS AB
# 2011 SCORING GUIDELINES

## Question 4

The continuous function $f$ is defined on the interval $-4 \le x \le 3$. The graph of $f$ consists of two quarter circles and one line segment, as shown in the figure above.




Graph of $f$

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the $x$-coordinate of the point at which $g$ has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of $x$ on the interval $-4 < x < 3$ for which the graph of $g$ has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of $f$ on the interval $-4 \le x \le 3$. There is no point $c$, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

---

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = -6 - \frac{9\pi}{4}$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3) = 2$

$3 : \begin{cases} 1 : g(-3) \\ 1 : g'(x) \\ 1 : g'(-3) \end{cases}$

(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$.

$g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$.

Therefore $g$ has an absolute maximum at $x = \frac{5}{2}$.

$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : \text{identifies interior candidate} \\ 1 : \text{answer with justification} \end{cases}$

(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of $g$ has a point of inflection at $x = 0$.

1 : answer with reason

(d) The average rate of change of $f$ on the interval $-4 \le x \le 3$ is $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$.

To apply the Mean Value Theorem, $f$ must be differentiable at each point in the interval $-4 < x < 3$. However, $f$ is not differentiable at $x = -3$ and $x = 0$.

$2 : \begin{cases} 1 : \text{average rate of change} \\ 1 : \text{explanation} \end{cases}$

### Question 4

Consider a differentiable function $f$ having domain all positive real numbers, and for which it is known that $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

(a) Find the $x$-coordinate of the critical point of $f$. Determine whether the point is a relative maximum, a relative minimum, or neither for the function $f$. Justify your answer.

(b) Find all intervals on which the graph of $f$ is concave down. Justify your answer.

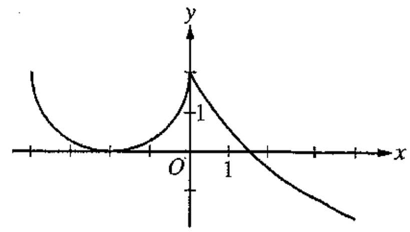
(c) Given that $f(1) = 2$, determine the function $f$.

| | |
|---|---|
| (a) $f'(x) = 0$ at $x = 4$<br>$f'(x) > 0$ for $0 < x < 4$<br>$f'(x) < 0$ for $x > 4$<br>Therefore $f$ has a relative maximum at $x = 4$. | 3 : { 1 : $x = 4$<br>1 : relative maximum<br>1 : justification |
| (b) $f''(x) = -x^{-3} + (4 - x)(-3x^{-4})$<br>$= -x^{-3} - 12x^{-4} + 3x^{-3}$<br>$= 2x^{-4}(x - 6)$<br>$= \dfrac{2(x-6)}{x^4}$<br>$f''(x) < 0$ for $0 < x < 6$<br>The graph of $f$ is concave down on the interval $0 < x < 6$. | 3 : { 2 : $f''(x)$<br>1 : answer with justification |
| (c) $f(x) = 2 + \int_1^x (4t^{-3} - t^{-2})\,dt$<br>$= 2 + \left[-2t^{-2} + t^{-1}\right]_{t=1}^{t=x}$<br>$= 3 - 2x^{-2} + x^{-1}$ | 3 : { 1 : integral<br>1 : antiderivative<br>1 : answer |

## Question 6

The derivative of a function $f$ is defined by

$$f'(x) = \begin{cases} g(x) & \text{for } -4 \le x \le 0 \\ 5e^{-x/3} - 3 & \text{for } \ 0 < x \le 4 \end{cases}.$$




Graph of $f'$

The graph of the continuous function $f'$, shown in the figure above, has $x$-intercepts at $x = -2$ and $x = 3\ln\left(\frac{5}{3}\right)$. The graph of $g$ on $-4 \le x \le 0$ is a semicircle, and $f(0) = 5$.

(a) For $-4 < x < 4$, find all values of $x$ at which the graph of $f$ has a point of inflection. Justify your answer.

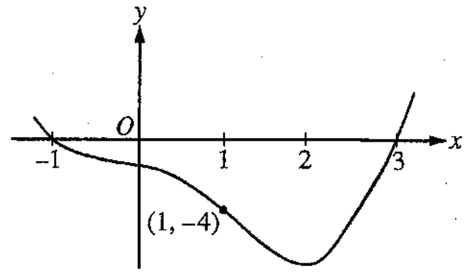
(b) Find $f(-4)$ and $f(4)$.

(c) For $-4 \le x \le 4$, find the value of $x$ at which $f$ has an absolute maximum. Justify your answer.

| | |
|---|---|
| (a) $f'$ changes from decreasing to increasing at $x = -2$ and from increasing to decreasing at $x = 0$. Therefore, the graph of $f$ has points of inflection at $x = -2$ and $x = 0$. | $2 : \begin{cases} 1 : \text{identifies } x = -2 \text{ or } x = 0 \\ 1 : \text{answer with justification} \end{cases}$ |
| (b) $f(-4) = 5 + \int_0^{-4} g(x)\,dx$ $= 5 - (8 - 2\pi) = 2\pi - 3$ <br> $f(4) = 5 + \int_0^4 \left(5e^{-x/3} - 3\right) dx$ $= 5 + \left(-15e^{-x/3} - 3x\right)\Big\|_{x=0}^{x=4}$ $= 8 - 15e^{-4/3}$ | $5 : \begin{cases} 2 : f(-4) \\ \quad 1 : \text{integral} \\ \quad 1 : \text{value} \\ 3 : f(4) \\ \quad 1 : \text{integral} \\ \quad 1 : \text{antiderivative} \\ \quad 1 : \text{value} \end{cases}$ |
| (c) Since $f'(x) > 0$ on the intervals $-4 < x < -2$ and $-2 < x < 3\ln\left(\frac{5}{3}\right)$, $f$ is increasing on the interval $-4 \le x \le 3\ln\left(\frac{5}{3}\right)$. <br> Since $f'(x) < 0$ on the interval $3\ln\left(\frac{5}{3}\right) < x < 4$, $f$ is decreasing on the interval $3\ln\left(\frac{5}{3}\right) \le x \le 4$. <br> Therefore, $f$ has an absolute maximum at $x = 3\ln\left(\frac{5}{3}\right)$. | $2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$ |

## Question 5

Let $f$ be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of $f'$, the derivative of $f$, is shown above. The graph of $f'$ crosses the $x$-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let $g$ be the function given by $g(x) = e^{f(x)}$.




Graph of $f'$

(a) Write an equation for the line tangent to the graph of $g$ at $x = 1$.

(b) For $-1.2 < x < 3.2$, find all values of $x$ at which $g$ has a local maximum. Justify your answer.

(c) The second derivative of $g$ is $g''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

(d) Find the average rate of change of $g'$, the derivative of $g$, over the interval $[1, 3]$.

---

| | |
|---|---|
| (a) $g(1) = e^{f(1)} = e^2$<br>$g'(x) = e^{f(x)}f'(x)$, $g'(1) = e^{f(1)}f'(1) = -4e^2$<br>The tangent line is given by $y = e^2 - 4e^2(x-1)$. | $3 : \begin{cases} 1 : g'(x) \\ 1 : g(1) \text{ and } g'(1) \\ 1 : \text{tangent line equation} \end{cases}$ |
| (b) $g'(x) = e^{f(x)}f'(x)$<br>$e^{f(x)} > 0$ for all $x$<br>So, $g'$ changes from positive to negative only when $f'$ changes from positive to negative. This occurs at $x = -1$ only. Thus, $g$ has a local maximum at $x = -1$. | $2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$ |
| (c) $g''(-1) = e^{f(-1)}\left[(f'(-1))^2 + f''(-1)\right]$<br>$e^{f(-1)} > 0$ and $f'(-1) = 0$<br>Since $f'$ is decreasing on a neighborhood of $-1$, $f''(-1) < 0$. Therefore, $g''(-1) < 0$. | $2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$ |
| (d) $\dfrac{g'(3) - g'(1)}{3 - 1} = \dfrac{e^{f(3)}f'(3) - e^{f(1)}f'(1)}{2} = 2e^2$ | $2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer} \end{cases}$ |

# AP® CALCULUS AB
# 2008 SCORING GUIDELINES

## Question 6

Let $f$ be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$. The derivative of $f$ is given by $f'(x) = \dfrac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of $f$ at $x = e^2$.

(b) Find the $x$-coordinate of the critical point of $f$. Determine whether this point is a relative minimum, a relative maximum, or neither for the function $f$. Justify your answer.

(c) The graph of the function $f$ has exactly one point of inflection. Find the $x$-coordinate of this point.

(d) Find $\lim\limits_{x \to 0^+} f(x)$.

---

(a) $f(e^2) = \dfrac{\ln e^2}{e^2} = \dfrac{2}{e^2}$, $f'(e^2) = \dfrac{1 - \ln e^2}{\left(e^2\right)^2} = -\dfrac{1}{e^4}$

An equation for the tangent line is $y = \dfrac{2}{e^2} - \dfrac{1}{e^4}\left(x - e^2\right)$.

$2 : \begin{cases} 1 : f(e^2) \text{ and } f'(e^2) \\ 1 : \text{answer} \end{cases}$

(b) $f'(x) = 0$ when $x = e$. The function $f$ has a relative maximum at $x = e$ because $f'(x)$ changes from positive to negative at $x = e$.

$3 : \begin{cases} 1 : x = e \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$

(c) $f''(x) = \dfrac{-\frac{1}{x}x^2 - (1 - \ln x)2x}{x^4} = \dfrac{-3 + 2\ln x}{x^3}$ for all $x > 0$

$f''(x) = 0$ when $-3 + 2\ln x = 0$

$x = e^{3/2}$

The graph of $f$ has a point of inflection at $x = e^{3/2}$ because $f''(x)$ changes sign at $x = e^{3/2}$.

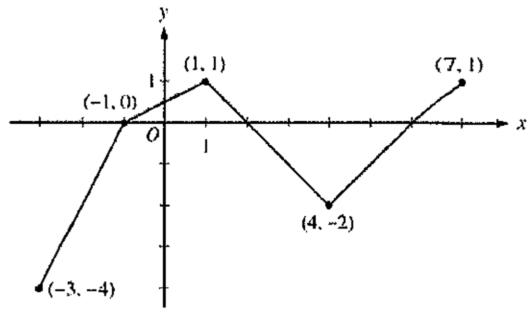
$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer} \end{cases}$

(d) $\lim\limits_{x \to 0^+} \dfrac{\ln x}{x} = -\infty$ or Does Not Exist

1 : answer

## Question 5

Let $g$ be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function $g'$, the derivative of $g$, is shown above for $-3 \le x \le 7$.




Graph of $g'$

(a) Find the $x$-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of $g$ on the interval $-3 \le x \le 7$. Justify your answer.

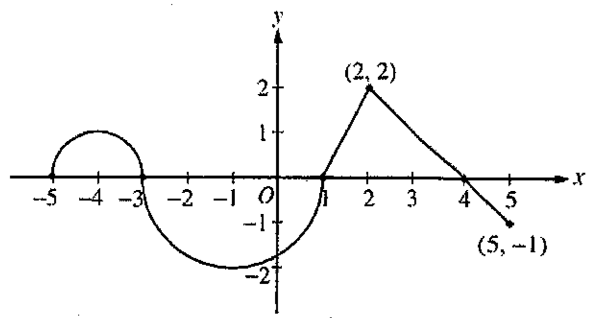
(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of $c$, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

---

| | |
|---|---|
| (a) $g'$ changes from increasing to decreasing at $x = 1$; $g'$ changes from decreasing to increasing at $x = 4$. <br><br> Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$. | $2: \begin{cases} 1: x\text{-values} \\ 1: \text{justification} \end{cases}$ |
| (b) The only sign change of $g'$ from positive to negative in the interval is at $x = 2$. <br><br> $g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$ <br> $g(2) = 5$ <br> $g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$ <br><br> The maximum value of $g$ for $-3 \le x \le 7$ is $\frac{15}{2}$. | $3: \begin{cases} 1: \text{identifies } x = 2 \text{ as a candidate} \\ 1: \text{considers endpoints} \\ 1: \text{maximum value and justification} \end{cases}$ |
| (c) $\dfrac{g(7) - g(-3)}{7 - (-3)} = \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = -\dfrac{3}{5}$ | $2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$ |
| (d) $\dfrac{g'(7) - g'(-3)}{7 - (-3)} = \dfrac{1 - (-4)}{10} = \dfrac{1}{2}$ <br> No, the MVT does *not* guarantee the existence of a value $c$ with the stated properties because $g'$ is not differentiable for at least one point in $-3 < x < 7$. | $2: \begin{cases} 1: \text{average value of } g'(x) \\ 1: \text{answer "No" with reason} \end{cases}$ |

### Question 4



Graph of $f'$

Let $f$ be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of $f'$, the derivative of $f$, consists of two semicircles and two line segments, as shown above.

(a) For $-5 < x < 5$, find all values $x$ at which $f$ has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values $x$ at which the graph of $f$ has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of $f$ is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

| | |
|---|---|
| (a) $f'(x) = 0$ at $x = -3,\ 1,\ 4$<br>$f'$ changes from positive to negative at $-3$ and 4.<br>Thus, $f$ has a relative maximum at $x = -3$ and at $x = 4$. | $2: \begin{cases} 1: x\text{-values} \\ 1: \text{justification} \end{cases}$ |
| (b) $f'$ changes from increasing to decreasing, or vice versa, at $x = -4,\ -1,$ and 2. Thus, the graph of $f$ has points of inflection when $x = -4,\ -1,$ and 2. | $2: \begin{cases} 1: x\text{-values} \\ 1: \text{justification} \end{cases}$ |
| (c) The graph of $f$ is concave up with positive slope where $f'$ is increasing and positive: $-5 < x < -4$ and $1 < x < 2$. | $2: \begin{cases} 1: \text{intervals} \\ 1: \text{explanation} \end{cases}$ |
| (d) Candidates for the absolute minimum are where $f'$ changes from negative to positive (at $x = 1$) and at the endpoints ($x = -5, 5$).<br>$f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi > 3$<br>$f(1) = 3$<br>$f(5) = 3 + \int_1^{5} f'(x)\,dx = 3 + \frac{3 \cdot 2}{2} - \frac{1}{2} > 3$<br>The absolute minimum value of $f$ on $[-5, 5]$ is $f(1) = 3$. | $3: \begin{cases} 1: \text{identifies } x = 1 \text{ as a candidate} \\ 1: \text{considers endpoints} \\ 1: \text{value and explanation} \end{cases}$ |

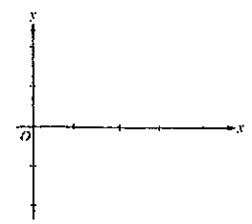
## Question 4

| $x$ | 0 | $0 < x < 1$ | 1 | $1 < x < 2$ | 2 | $2 < x < 3$ | 3 | $3 < x < 4$ |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | –1 | Negative | 0 | Positive | 2 | Positive | 0 | Negative |
| $f'(x)$ | 4 | Positive | 0 | Positive | DNE | Negative | –3 | Negative |
| $f''(x)$ | –2 | Negative | 0 | Positive | DNE | Negative | 0 | Positive |

Let $f$ be a function that is continuous on the interval $[0, 4)$. The function $f$ is twice differentiable except at $x = 2$. The function $f$ and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of $f$ do not exist at $x = 2$.

(a) For $0 < x < 4$, find all values of $x$ at which $f$ has a relative extremum. Determine whether $f$ has a relative maximum or a relative minimum at each of these values. Justify your answer.

(b) On the axes provided, sketch the graph of a function that has all the characteristics of $f$. **(Note: Use the axes provided in the pink test booklet.)**



(c) Let $g$ be the function defined by $g(x) = \int_1^x f(t)\,dt$ on the open interval $(0, 4)$. For $0 < x < 4$, find all values of $x$ at which $g$ has a relative extremum. Determine whether $g$ has a relative maximum or a relative minimum at each of these values. Justify your answer.

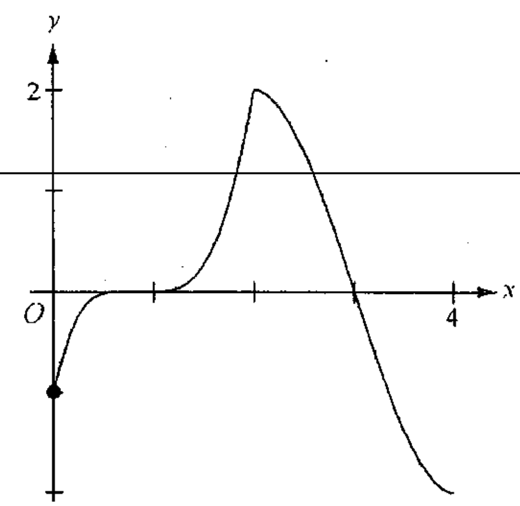
(d) For the function $g$ defined in part (c), find all values of $x$, for $0 < x < 4$, at which the graph of $g$ has a point of inflection. Justify your answer.

---

(a) $f$ has a relative maximum at $x = 2$ because $f'$ changes from positive to negative at $x = 2$.

$2 : \begin{cases} 1 : \text{relative extremum at } x = 2 \\ 1 : \text{relative maximum with justification} \end{cases}$

(b)




$2 : \begin{cases} 1 : \text{points at } x = 0, 1, 2, 3 \\ \quad \text{and behavior at } (2, 2) \\ 1 : \text{appropriate increasing/decreasing} \\ \quad \text{and concavity behavior} \end{cases}$

(c) $g'(x) = f(x) = 0$ at $x = 1, 3$.
$g'$ changes from negative to positive at $x = 1$ so $g$ has a relative minimum at $x = 1$. $g'$ changes from positive to negative at $x = 3$ so $g$ has a relative maximum at $x = 3$.

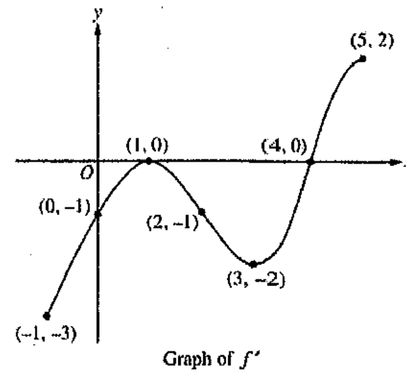
$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{critical points} \\ 1 : \text{answer with justification} \end{cases}$

(d) The graph of $g$ has a point of inflection at $x = 2$ because $g'' = f'$ changes sign at $x = 2$.

$2 : \begin{cases} 1 : x = 2 \\ 1 : \text{answer with justification} \end{cases}$

## Question 4

The figure above shows the graph of $f'$, the derivative of the function $f$, on the closed interval $-1 \le x \le 5$. The graph of $f'$ has horizontal tangent lines at $x = 1$ and $x = 3$. The function $f$ is twice differentiable with $f(2) = 6$.



Graph of $f'$

(a) Find the $x$-coordinate of each of the points of inflection of the graph of $f$. Give a reason for your answer.

(b) At what value of $x$ does $f$ attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of $x$ does $f$ attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

(c) Let $g$ be the function defined by $g(x) = xf(x)$. Find an equation for the line tangent to the graph of $g$ at $x = 2$.

---

| | |
|---|---|
| (a) $x = 1$ and $x = 3$ because the graph of $f'$ changes from increasing to decreasing at $x = 1$, and changes from decreasing to increasing at $x = 3$. | $2: \begin{cases} 1: x = 1, x = 3 \\ 1: \text{reason} \end{cases}$ |
| (b) The function $f$ decreases from $x = -1$ to $x = 4$, then increases from $x = 4$ to $x = 5$. Therefore, the absolute minimum value for $f$ is at $x = 4$. The absolute maximum value must occur at $x = -1$ or at $x = 5$. $f(5) - f(-1) = \int_{-1}^{5} f'(t)\,dt < 0$ Since $f(5) < f(-1)$, the absolute maximum value occurs at $x = -1$. | $4: \begin{cases} 1: \text{indicates } f \text{ decreases then increases} \\ 1: \text{eliminates } x = 5 \text{ for maximum} \\ 1: \text{absolute minimum at } x = 4 \\ 1: \text{absolute maximum at } x = -1 \end{cases}$ |
| (c) $g'(x) = f(x) + xf'(x)$ <br> $g'(2) = f(2) + 2f'(2) = 6 + 2(-1) = 4$ <br> $g(2) = 2f(2) = 12$ <br> Tangent line is $y = 4(x - 2) + 12$ | $3: \begin{cases} 2: g'(x) \\ 1: \text{tangent line} \end{cases}$ |

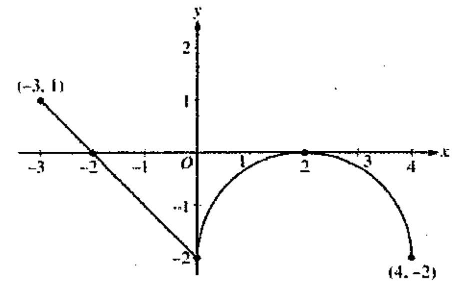






73

### Question 4




Graph of $f'$

Let $f$ be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of $f'$, the derivative of $f$, consists of one line segment and a semicircle, as shown above.

(a) On what intervals, if any, is $f$ increasing? Justify your answer.

(b) Find the $x$-coordinate of each point of inflection of the graph of $f$ on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of $f$ at the point $(0,3)$.

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

| | |
|---|---|
| (a) The function $f$ is increasing on $[-3,-2]$ since $f' > 0$ for $-3 \le x < -2$. | $2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$ |
| (b) $x = 0$ and $x = 2$<br>$f'$ changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$ | $2 : \begin{cases} 1 : x = 0 \text{ and } x = 2 \text{ only} \\ 1 : \text{justification} \end{cases}$ |
| (c) $f'(0) = -2$<br>Tangent line is $y = -2x + 3$. | 1 : equation |
| (d) $f(0) - f(-3) = \int_{-3}^{0} f'(t)\,dt = \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$<br>$f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$<br>$f(4) - f(0) = \int_{0}^{4} f'(t)\,dt = -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$<br>$f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$ | 4 : 1 : $\pm\left(\frac{1}{2} - 2\right)$ (difference of areas of triangles)<br>1 : answer for $f(-3)$ using FTC<br>1 : $\pm\left(8 - \frac{1}{2}(2)^2\pi\right)$ (area of rectangle − area of semicircle)<br>1 : answer for $f(4)$ using FTC |

## Question 6

Let $f$ be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where $k$ is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant $k$ does $f$ have a critical point at $x = 1$? For this value of $k$, determine whether $f$ has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant $k$, the graph of $f$ has a point of inflection on the $x$-axis. Find this value of $k$.

---

| | |
|---|---|
| (a) $f'(x) = \dfrac{k}{2\sqrt{x}} - \dfrac{1}{x}$<br><br>$f''(x) = -\dfrac{1}{4}kx^{-3/2} + x^{-2}$ | $2 : \begin{cases} 1 : f'(x) \\ 1 : f''(x) \end{cases}$ |
| (b) $f'(1) = \dfrac{1}{2}k - 1 = 0 \Rightarrow k = 2$<br><br>When $k = 2$, $f'(1) = 0$ and $f''(1) = -\dfrac{1}{2} + 1 > 0$.<br><br>$f$ has a relative minimum value at $x = 1$ by the Second Derivative Test. | $4 : \begin{cases} 1 : \text{sets } f'(1) = 0 \text{ or } f'(x) = 0 \\ 1 : \text{solves for } k \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$ |
| (c) At this inflection point, $f''(x) = 0$ and $f(x) = 0$.<br><br>$f''(x) = 0 \Rightarrow \dfrac{-k}{4x^{3/2}} + \dfrac{1}{x^2} = 0 \Rightarrow k = \dfrac{4}{\sqrt{x}}$<br><br>$f(x) = 0 \Rightarrow k\sqrt{x} - \ln x = 0 \Rightarrow k = \dfrac{\ln x}{\sqrt{x}}$<br><br>Therefore, $\dfrac{4}{\sqrt{x}} = \dfrac{\ln x}{\sqrt{x}}$<br>$\Rightarrow 4 = \ln x$<br>$\Rightarrow x = e^4$<br>$\Rightarrow k = \dfrac{4}{e^2}$ | $3 : \begin{cases} 1 : f''(x) = 0 \text{ or } f(x) = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{answer} \end{cases}$ |

## Question 2

The function $g$ is defined for $x > 0$ with $g(1) = 2$, $g'(x) = \sin\left(x + \frac{1}{x}\right)$, and $g''(x) = \left(1 - \frac{1}{x^2}\right)\cos\left(x + \frac{1}{x}\right)$.

(a) Find all values of $x$ in the interval $0.12 \le x \le 1$ at which the graph of $g$ has a horizontal tangent line.

(b) On what subintervals of $(0.12, 1)$, if any, is the graph of $g$ concave down? Justify your answer.

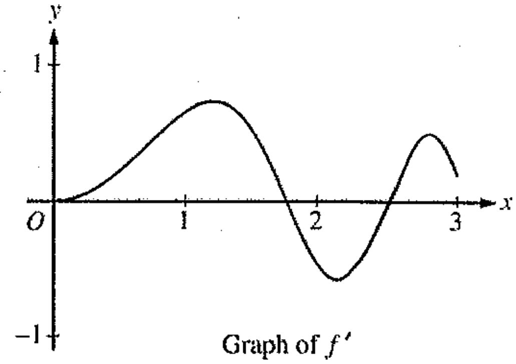
(c) Write an equation for the line tangent to the graph of $g$ at $x = 0.3$.

(d) Does the line tangent to the graph of $g$ at $x = 0.3$ lie above or below the graph of $g$ for $0.3 < x < 1$? Why?

| | |
|---|---|
| (a) The graph of $g$ has a horizontal tangent line when $g'(x) = 0$. This occurs at $x = 0.163$ and $x = 0.359$. | $2 : \begin{cases} 1 : \text{sets } g'(x) = 0 \\ 1 : \text{answer} \end{cases}$ |
| (b) $g''(x) = 0$ at $x = 0.129458$ and $x = 0.222734$<br><br>The graph of $g$ is concave down on $(0.1295, 0.2227)$ because $g''(x) < 0$ on this interval. | $2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$ |
| (c) $g'(0.3) = -0.472161$<br>$g(0.3) = 2 + \int_1^{0.3} g'(x)\,dx = 1.546007$<br><br>An equation for the line tangent to the graph of $g$ is $y = 1.546 - 0.472(x - 0.3)$. | $4 : \begin{cases} 1 : g'(0.3) \\ 1 : \text{integral expression} \\ 1 : g(0.3) \\ 1 : \text{equation} \end{cases}$ |
| (d) $g''(x) > 0$ for $0.3 < x < 1$<br><br>Therefore the line tangent to the graph of $g$ at $x = 0.3$ lies below the graph of $g$ for $0.3 < x < 1$. | 1 : answer with reason |

## Question 2

Let $f$ be the function defined for $x \geq 0$ with $f(0) = 5$ and $f'$, the first derivative of $f$, given by $f'(x) = e^{(-x/4)} \sin\left(x^2\right)$. The graph of $y = f'(x)$ is shown above.




Graph of $f'$

(a) Use the graph of $f'$ to determine whether the graph of $f$ is concave up, concave down, or neither on the interval $1.7 < x < 1.9$. Explain your reasoning.

(b) On the interval $0 \leq x \leq 3$, find the value of $x$ at which $f$ has an absolute maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of $f$ at $x = 2$.

---

(a) On the interval $1.7 < x < 1.9$, $f'$ is decreasing and thus $f$ is concave down on this interval.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(b) $f'(x) = 0$ when $x = 0, \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \ldots$
On $[0, 3]$ $f'$ changes from positive to negative only at $\sqrt{\pi}$. The absolute maximum must occur at $x = \sqrt{\pi}$ or at an endpoint.

$f(0) = 5$

$f\left(\sqrt{\pi}\right) = f(0) + \int_0^{\sqrt{\pi}} f'(x)\, dx = 5.67911$

$f(3) = f(0) + \int_0^3 f'(x)\, dx = 5.57893$

This shows that $f$ has an absolute maximum at $x = \sqrt{\pi}$.

3 :
- 1 : identifies $\sqrt{\pi}$ and 3 as candidates
  - or -
  indicates that the graph of $f$ increases, decreases, then increases
- 1 : justifies $f\left(\sqrt{\pi}\right) > f(3)$
- 1 : answer

(c) $f(2) = f(0) + \int_0^2 f'(x)\, dx = 5.62342$

$f'(2) = e^{-0.5} \sin(4) = -0.45902$

$y - 5.623 = (-0.459)(x - 2)$

4 :
- 2 : $f(2)$ expression
  - 1 : integral
  - 1 : including $f(0)$ term
- 1 : $f'(2)$
- 1 : equation

# Optimization

97 82. If $y = 2x - 8$, what is the minimum value of the product $xy$ ?

(A) $-16$ (B) $-8$ (C) $-4$ (D) $0$ (E) $2$

93 BC 36. Consider all right circular cylinders for which the sum of the height and circumference is 30 centimeters. What is the radius of the one with maximum volume?

(A) 3 cm (B) 10 cm (C) 20 cm (D) $\frac{30}{\pi^2}$ cm (E) $\frac{10}{\pi}$ cm

88 45. The volume of a cylindrical tin can with a top and a bottom is to be $16\pi$ cubic inches. If a minimum amount of tin is to be used to construct the can, what must be the height, in inches, of the can?

(A) $2\sqrt[3]{2}$ (B) $2\sqrt{2}$ (C) $2\sqrt[3]{4}$ (D) 4 (E) 8

69 BC 11. The point on the curve $x^2 + 2y = 0$ that is nearest the point $\left(0, -\frac{1}{2}\right)$ occurs where $y$ is

(A) $\frac{1}{2}$
(B) $0$
(C) $-\frac{1}{2}$
(D) $-1$
(E) none of the above

73 39. The point on the curve $2y = x^2$ nearest to $(4,1)$ is

(A) $(0,0)$ (B) $(2,2)$ (C) $(\sqrt{2},1)$ (D) $(2\sqrt{2},4)$ (E) $(4,8)$

88 BC 45. What is the area of the largest rectangle that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$?

(A) $6\sqrt{2}$ (B) 12 (C) 24 (D) $24\sqrt{2}$ (E) 36

# Linear Approximations

18, C 92. Let $f$ be the function given by $f(x) = x^2 - 2x + 3$. The tangent line to the graph of $f$ at $x = 2$ is used to approximate values of $f(x)$. Which of the following is the greatest value of $x$ for which the error resulting from this tangent line approximation is less than 0.5?

(A) 2.4 (B) 2.5 (C) 2.6 (D) 2.7 (E) 2.8

69 36. The approximate value of $y = \sqrt{4 + \sin x}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$, is

(A) 2.00 (B) 2.03 (C) 2.06 (D) 2.12 (E) 2.24

69 37. Which is the best of the following polynomial approximations to $\cos 2x$ near $x = 0$?

(A) $1 + \frac{x}{2}$ (B) $1 + x$ (C) $1 - \frac{x^2}{2}$ (D) $1 - 2x^2$ (E) $1 - 2x + x^2$

97 14. Let $f$ be a differentiable function such that $f(3) = 2$ and $f'(3) = 5$. If the tangent line to the graph of $f$ at $x = 3$ is used to find an approximation to a zero of $f$, that approximation is

(A) 0.4 (B) 0.5 (C) 2.6 (D) 3.4 (E) 5.5

93 45. If Newton's method is used to approximate the real root of $x^3 + x - 1 = 0$, then a first approximation $x_1 = 1$ would lead to a third approximation of $x_3 =$

(A) 0.682 (B) 0.686 (C) 0.694 (D) 0.750 (E) 1.637

93 BC 25. Consider the curve in the $xy$-plane represented by $x = e^t$ and $y = te^{-t}$ for $t \geq 0$. The slope of the line tangent to the curve at the point where $x = 3$ is

(A) 20.086 (B) 0.342 (C) −0.005 (D) −0.011 (E) −0.033

73 44. For small values of $h$, the function $\sqrt[4]{16+h}$ is best approximated by which of the following?

(A) $4+\frac{h}{32}$ (B) $2+\frac{h}{32}$ (C) $\frac{h}{32}$

(D) $4-\frac{h}{32}$ (E) $2-\frac{h}{32}$



80

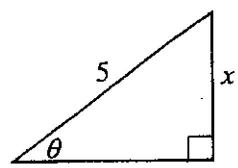
# Related Rates

93 26. The radius $r$ of a sphere is increasing at the uniform rate of 0.3 inches per second. At the instant when the surface area $S$ becomes $100\pi$ square inches, what is the rate of increase, in cubic inches per second, in the volume $V$? $\left(S = 4\pi r^2 \text{ and } V = \frac{4}{3}\pi r^3\right)$

(A) $10\pi$ (B) $12\pi$ (C) $22.5\pi$ (D) $25\pi$ (E) $30\pi$

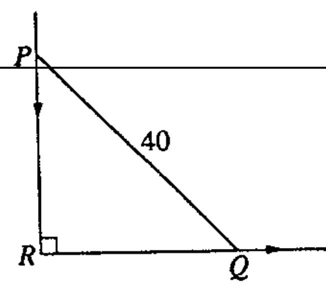
93 39. The radius of a circle is increasing at a nonzero rate, and at a certain instant, the rate of increase in the area of the circle is numerically equal to the rate of increase in its circumference. At this instant, the radius of the circle is

(A) $\frac{1}{\pi}$ (B) $\frac{1}{2}$ (C) $\frac{2}{\pi}$ (D) 1 (E) 2



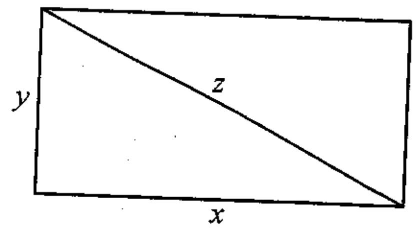

97 BC 23. In the triangle shown above, if $\theta$ increases at a constant rate of 3 radians per minute, at what rate is $x$ increasing in units per minute when $x$ equals 3 units?

(A) 3 (B) $\frac{15}{4}$ (C) 4 (D) 9 (E) 12




93 BC 34. In the figure above, $PQ$ represents a 40-foot ladder with end $P$ against a vertical wall and end $Q$ on level ground. If the ladder is slipping down the wall, what is the distance $RQ$ at the instant when $Q$ is moving along the ground $\frac{3}{4}$ as fast as $P$ is moving down the wall?

(A) $\frac{6}{5}\sqrt{10}$ (B) $\frac{8}{5}\sqrt{10}$ (C) $\frac{80}{\sqrt{7}}$ (D) 24 (E) 32




88 40. The sides of the rectangle above increase in such a way that $\frac{dz}{dt} = 1$ and $\frac{dx}{dt} = 3\frac{dy}{dt}$. At the instant when $x = 4$ and $y = 3$, what is the value of $\frac{dx}{dt}$?

(A) $\frac{1}{3}$ (B) 1 (C) 2 (D) $\sqrt{5}$ (E) 5

---

88 BC 37. A person 2 meters tall walks directly away from a streetlight that is 8 meters above the ground. If the person is walking at a constant rate and the person's shadow is lengthening at the rate of $\frac{4}{9}$ meter per second, at what rate, in meters per second, is the person walking?

(A) $\frac{4}{27}$ (B) $\frac{4}{9}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$ (E) $\frac{16}{9}$

---

98 BC 78. The radius of a circle is decreasing at a constant rate of 0.1 centimeter per second. In terms of the circumference $C$, what is the rate of change of the area of the circle, in square centimeters per second?

(A) $-(0.2)\pi C$

(B) $-(0.1)C$

(C) $-\frac{(0.1)C}{2\pi}$

(D) $(0.1)^2 C$

(E) $(0.1)^2 \pi C$

93 34. The top of a 25-foot ladder is sliding down a vertical wall at a constant rate of 3 feet per minute. When the top of the ladder is 7 feet from the ground, what is the rate of change of the distance between the bottom of the ladder and the wall?

(A) $-\frac{7}{8}$ feet per minute

(B) $-\frac{7}{24}$ feet per minute

(C) $\frac{7}{24}$ feet per minute

(D) $\frac{7}{8}$ feet per minute

(E) $\frac{21}{25}$ feet per minute

69 AB 9. When the area in square units of an expanding circle is increasing twice as fast as its radius in linear units, the radius is

(A) $\frac{1}{4\pi}$ (B) $\frac{1}{4}$ (C) $\frac{1}{\pi}$ (D) 1 (E) $\pi$

85 31. The volume of a cone of radius $r$ and height $h$ is given by $V = \frac{1}{3}\pi r^2 h$. If the radius and the height both increase at a constant rate of $\frac{1}{2}$ centimeter per second, at what rate, in cubic centimeters per second, is the volume increasing when the height is 9 centimeters and the radius is 6 centimeters?

(A) $\frac{1}{2}\pi$ (B) $10\pi$ (C) $24\pi$ (D) $54\pi$ (E) $108\pi$

85 BC 22. The area of a circular region is increasing at a rate of $96\pi$ square meters per second. When the area of the region is $64\pi$ square meters, how fast, in meters per second, is the radius of the region increasing?

(A) 6 (B) 8 (C) 16 (D) $4\sqrt{3}$ (E) $12\sqrt{3}$

98 90. If the base $b$ of a triangle is increasing at a rate of 3 inches per minute while its height $h$ is decreasing at a rate of 3 inches per minute, which of the following must be true about the area $A$ of the triangle?

(A) $A$ is always increasing.
(B) $A$ is always decreasing.
(C) $A$ is decreasing only when $b < h$.
(D) $A$ is decreasing only when $b > h$.
(E) $A$ remains constant.

97 81. A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the crossing and watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

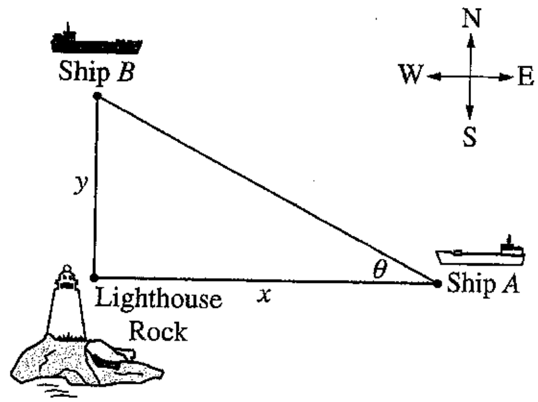
(A) 57.60 (B) 57.88 (C) 59.20 (D) 60.00 (E) 67.40

97 86. Let $f(x) = \sqrt{x}$. If the rate of change of $f$ at $x = c$ is twice its rate of change at $x = 1$, then $c =$

(A) $\frac{1}{4}$ (B) 1 (C) 4 (D) $\frac{1}{\sqrt{2}}$ (E) $\frac{1}{2\sqrt{2}}$

98 BC 20. When $x = 8$, the rate at which $\sqrt[3]{x}$ is increasing is $\frac{1}{k}$ times the rate at which $x$ is increasing. What is the value of $k$?

(A) 3 (B) 4 (C) 6 (D) 8 (E) 12



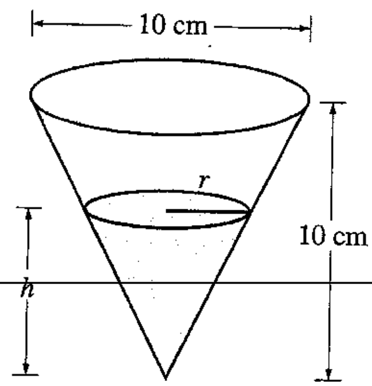

2-B

6. Ship $A$ is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship $B$ is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let $x$ be the distance between Ship $A$ and Lighthouse Rock at time $t$, and let $y$ be the distance between Ship $B$ and Lighthouse Rock at time $t$, as shown in the figure above.

   (a) Find the distance, in kilometers, between Ship $A$ and Ship $B$ when $x = 4$ km and $y = 3$ km.

   (b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

   (c) Let $\theta$ be the angle shown in the figure. Find the rate of change of $\theta$, in radians per hour, when $x = 4$ km and $y = 3$ km.




2-A

5. A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth $h$ is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

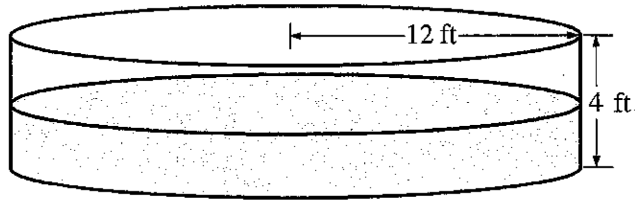
   (Note: The volume of a cone of height $h$ and radius $r$ is given by $V = \frac{1}{3}\pi r^2 h$.)

   (a) Find the volume $V$ of water in the container when $h = 5$ cm. Indicate units of measure.

   (b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

   (c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

| $t$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $P(t)$ | 0 | 46 | 53 | 57 | 60 | 62 | 63 |




10-B

3. The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of $t$. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume $V$ of a cylinder with radius $r$ and height $h$ is given by $V = \pi r^2 h$.)

   (a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

   (b) Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

   (c) Use the results from parts (a) and (b) to approximate the volume of water in the pool at time $t = 12$ hours. Round your answer to the nearest cubic foot.

   (d) Find the rate at which the volume of water in the pool is increasing at time $t = 8$ hours. How fast is the water level in the pool rising at $t = 8$ hours? Indicate units of measure in both answers.

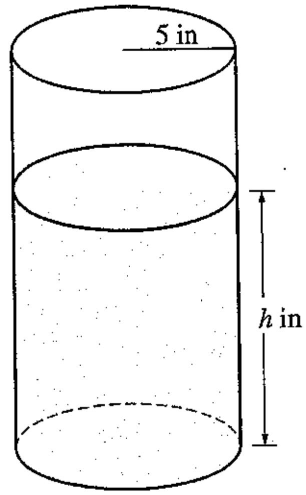
8-A

3. Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume $V$ of a right circular cylinder with radius $r$ and height $h$ is given by $V = \pi r^2 h$.)

   (a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

   (b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where $t$ is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time $t$ when the oil slick reaches its maximum volume. Justify your answer.

   (c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).




3-A

5. A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let $h$ be the depth of the coffee in the pot, measured in inches, where $h$ is a function of time $t$, measured in seconds. The volume $V$ of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume $V$ of a cylinder with radius $r$ and height $h$ is $V = \pi r^2 h$.)

(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for $h$ as a function of $t$.

(c) At what time $t$ is the coffeepot empty?



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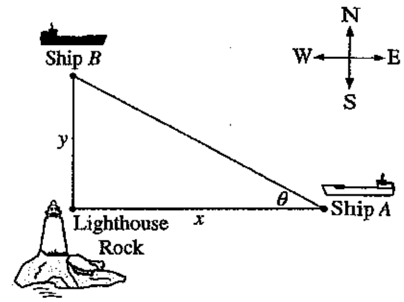
### Question 6

Ship $A$ is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship $B$ is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let $x$ be the distance between Ship $A$ and Lighthouse Rock at time $t$, and let $y$ be the distance between Ship $B$ and Lighthouse Rock at time $t$, as shown in the figure above.




(a) Find the distance, in kilometers, between Ship $A$ and Ship $B$ when $x = 4$ km and $y = 3$ km.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let $\theta$ be the angle shown in the figure. Find the rate of change of $\theta$, in radians per hour, when $x = 4$ km and $y = 3$ km.

---

(a) Distance $= \sqrt{3^2 + 4^2} = 5$ km

1 : answer

(b) $r^2 = x^2 + y^2$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or explicitly:

$$r = \sqrt{x^2 + y^2}$$

$$\frac{dr}{dt} = \frac{1}{2\sqrt{x^2 + y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

At $x = 4$, $y = 3$,

$$\frac{dr}{dt} = \frac{4(-15) + 3(10)}{5} = -6 \text{ km/hr}$$

4 $\begin{cases} 1 : \text{expression for distance} \\ 2 : \text{differentiation with respect to } t \\ \quad < -2 > \text{ chain rule error} \\ 1 : \text{evaluation} \end{cases}$

(c) $\tan\theta = \dfrac{y}{x}$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$$

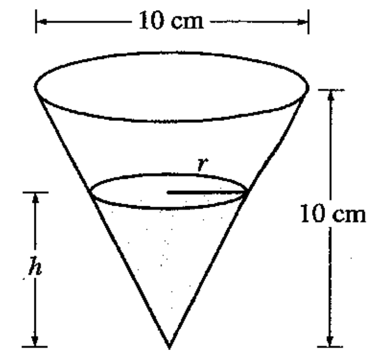
At $x = 4$ and $y = 3$, $\sec\theta = \dfrac{5}{4}$

$$\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4) - (-15)(3)}{16}\right)$$

$$= \frac{85}{25} = \frac{17}{5} \text{ radians/hr}$$

4 $\begin{cases} 1 : \text{expression for } \theta \text{ in terms of } x \text{ and } y \\ 2 : \text{differentiation with respect to } t \\ \quad < -2 > \text{ chain rule, quotient rule, or transcendental function error} \\ \quad \text{note: 0/2 if no trig or inverse trig function} \\ 1 : \text{evaluation} \end{cases}$

### Question 5




A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth $h$ is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(The volume of a cone of height $h$ and radius $r$ is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume $V$ of water in the container when $h = 5$ cm. Indicate units of measure.

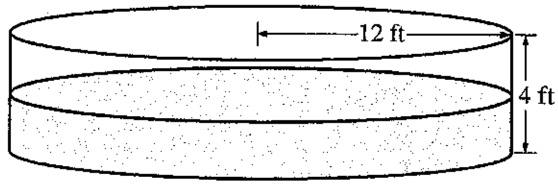
(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

| | |
|---|---|
| (a) When $h = 5$, $r = \frac{5}{2}$; $V(5) = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 5 = \frac{125}{12}\pi$ cm$^3$ | 1 : $V$ when $h = 5$ |
| (b) $\frac{r}{h} = \frac{5}{10}$, so $r = \frac{1}{2}h$<br>$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3$; $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$<br>$\left.\frac{dV}{dt}\right\vert_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi$ cm$^3$/hr<br>OR<br>$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$; $\frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$<br>$\left.\frac{dV}{dt}\right\vert_{h=5, r=\frac{5}{2}} = \frac{1}{3}\pi\left(\left(\frac{25}{4}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\right)$<br>$= -\frac{15}{8}\pi$ cm$^3$/hr | 5 { 1 : $r = \frac{1}{2}h$ in (a) or (b)<br>1 : { $V$ as a function of one variable in (a) or (b) OR $\frac{dr}{dt}$ }<br>2 : $\frac{dV}{dt}$<br>< −2 > chain rule or product rule error<br>1 : evaluation at $h = 5$ } |
| (c) $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt} = -\frac{3}{40}\pi h^2$<br>$= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10}\cdot$ area<br>The constant of proportionality is $-\frac{3}{10}$. | 2 { 1 : shows $\frac{dV}{dt} = k \cdot$ area<br>1 : identifies constant of proportionality } |
| **units of cm$^3$ in (a) and cm$^3$/hr in (b)** | **1 : correct units in (a) and (b)** |

### Question 3

| $t$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $P(t)$ | 0 | 46 | 53 | 57 | 60 | 62 | 63 |




The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of $t$. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume $V$ of a cylinder with radius $r$ and height $h$ is given by $V = \pi r^2 h$.)

(a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

(b) Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

(c) Use the results from parts (a) and (b) to approximate the volume of water in the pool at time $t = 12$ hours. Round your answer to the nearest cubic foot.

(d) Find the rate at which the volume of water in the pool is increasing at time $t = 8$ hours. How fast is the water level in the pool rising at $t = 8$ hours? Indicate units of measure in both answers.

---

(a) $\int_0^{12} P(t)\,dt \approx 46 \cdot 4 + 57 \cdot 4 + 62 \cdot 4 = 660 \text{ ft}^3$

$2 : \begin{cases} 1 : \text{midpoint sum} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^{12} R(t)\,dt = 225.594 \text{ ft}^3$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

---

(c) $1000 + \int_0^{12} P(t)\,dt - \int_0^{12} R(t)\,dt = 1434.406$

At time $t = 12$ hours, the volume of water in the pool is approximately $1434 \text{ ft}^3$.

1 : answer

(d) $V'(t) = P(t) - R(t)$

$V'(8) = P(8) - R(8) = 60 - 25e^{-0.4} = 43.241 \text{ or } 43.242 \text{ ft}^3/\text{hr}$

$V = \pi(12)^2 h$

$\frac{dV}{dt} = 144\pi \frac{dh}{dt}$

$\left.\frac{dh}{dt}\right|_{t=8} = \frac{1}{144\pi} \cdot \left.\frac{dV}{dt}\right|_{t=8} = 0.095 \text{ or } 0.096 \text{ ft/hr}$

$4 : \begin{cases} 1 : V'(8) \\ 1 : \text{equation relating } \frac{dV}{dt} \text{ and } \frac{dh}{dt} \\ 1 : \left.\frac{dh}{dt}\right|_{t=8} \\ 1 : \text{units of ft}^3/\text{hr and ft/hr} \end{cases}$

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# AP® CALCULUS AB
# 2008 SCORING GUIDELINES

## Question 3

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume $V$ of a right circular cylinder with radius $r$ and height $h$ is given by $V = \pi r^2 h$.)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where $t$ is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time $t$ when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

| Solution | Scoring |
|---|---|
| (a) When $r = 100$ cm and $h = 0.5$ cm, $\frac{dV}{dt} = 2000\ \text{cm}^3/\text{min}$ and $\frac{dr}{dt} = 2.5\ \text{cm}/\text{min}$.<br>$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$<br>$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$<br>$\frac{dh}{dt} = 0.038$ or $0.039$ cm/min | 4 : { 1 : $\frac{dV}{dt} = 2000$ and $\frac{dr}{dt} = 2.5$<br>2 : expression for $\frac{dV}{dt}$<br>1 : answer |
| (b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$.<br>This occurs when $t = 25$ minutes.<br>Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$, the oil slick reaches its maximum volume 25 minutes after the device begins working. | 3 : { 1 : $R(t) = 2000$<br>1 : answer<br>1 : justification |
| (c) The volume of oil, in $\text{cm}^3$, in the slick at time $t = 25$ minutes is given by $60,000 + \int_0^{25} (2000 - R(t))\, dt$. | 2 : { 1 : limits and initial condition<br>1 : integrand |





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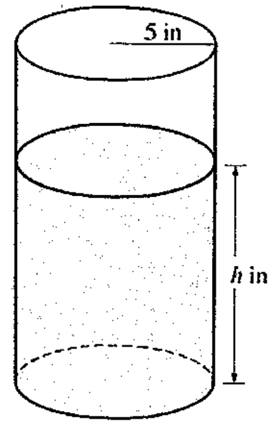
## Question 5

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let $h$ be the depth of the coffee in the pot, measured in inches, where $h$ is a function of time $t$, measured in seconds. The volume $V$ of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume $V$ of a cylinder with radius $r$ and height $h$ is $V = \pi r^2 h$.)




(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for $h$ as a function of $t$.

(c) At what time $t$ is the coffeepot empty?

---

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

3 : $\begin{cases} 1 : \frac{dV}{dt} = -5\pi\sqrt{h} \\ 1 : \text{computes } \frac{dV}{dt} \\ 1 : \text{shows result} \end{cases}$

(b) $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}\,dh = -\frac{1}{5}\,dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

5 : $\begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } h = 17 \\ \quad \text{when } t = 0 \\ 1 : \text{solves for } h \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer




92

# Piecewise Function FRQs

2011

6. Let $f$ be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that $f$ is continuous at $x = 0$.

(b) For $x \ne 0$, express $f'(x)$ as a piecewise-defined function. Find the value of $x$ for which $f'(x) = -3$.

(c) Find the average value of $f$ on the interval $[-1, 1]$.

3-A

6. Let $f$ be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is $f$ continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function $g$ is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where $k$ and $m$ are constants. If $g$ is differentiable at $x = 3$, what are the values of $k$ and $m$?

## AP® CALCULUS AB
## 2011 SCORING GUIDELINES

### Question 6

Let $f$ be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that $f$ is continuous at $x = 0$.

(b) For $x \ne 0$, express $f'(x)$ as a piecewise-defined function. Find the value of $x$ for which $f'(x) = -3$.

(c) Find the average value of $f$ on the interval $[-1, 1]$.

| | |
|---|---|
| (a) $\lim_{x\to 0^-}(1 - 2\sin x) = 1$<br>$\lim_{x\to 0^+} e^{-4x} = 1$<br>$f(0) = 1$<br>So, $\lim_{x\to 0} f(x) = f(0)$.<br>Therefore $f$ is continuous at $x = 0$. | 2 : analysis |
| (b) $f'(x) = \begin{cases} -2\cos x & \text{for } x < 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases}$<br>$-2\cos x \ne -3$ for all values of $x < 0$.<br>$-4e^{-4x} = -3$ when $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right) > 0$.<br>Therefore $f'(x) = -3$ for $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right)$. | $3 : \begin{cases} 2 : f'(x) \\ 1 : \text{value of } x \end{cases}$ |
| (c) $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$<br>$= \int_{-1}^{0}(1 - 2\sin x)\,dx + \int_{0}^{1} e^{-4x}\,dx$<br>$= \Big[x + 2\cos x\Big]_{x=-1}^{x=0} + \Big[-\frac{1}{4}e^{-4x}\Big]_{x=0}^{x=1}$<br>$= (3 - 2\cos(-1)) + \left(-\frac{1}{4}e^{-4} + \frac{1}{4}\right)$<br>Average value $= \frac{1}{2}\int_{-1}^{1} f(x)\,dx$<br>$= \frac{13}{8} - \cos(-1) - \frac{1}{8}e^{-4}$ | $4 : \begin{cases} 1 : \int_{-1}^{0}(1 - 2\sin x)\,dx \text{ and } \int_{0}^{1} e^{-4x}\,dx \\ 2 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$ |

## Question 6

Let $f$ be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is $f$ continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function $g$ is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where $k$ and $m$ are constants. If $g$ is differentiable at $x = 3$, what are the values of $k$ and $m$ ?

---

(a) $f$ is continuous at $x = 3$ because
$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = 2.$
Therefore, $\lim\limits_{x \to 3} f(x) = 2 = f(3).$

$2 : \begin{cases} 1 : \text{answers "yes" and equates the values of the left- and right-hand limits} \\ 1 : \text{explanation involving limits} \end{cases}$

(b)
$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$
$$= \frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x - \frac{1}{2}x^2\right)\Big|_3^5$$
$$= \left(\frac{16}{3} - \frac{2}{3}\right) + \left(\frac{25}{2} - \frac{21}{2}\right) = \frac{20}{3}$$

Average value: $\frac{1}{5}\int_0^5 f(x)\,dx = \frac{4}{3}$

$4 : \begin{cases} 1 : k\int_0^3 f(x)\,dx + k\int_3^5 f(x)\,dx \\ \quad (\text{where } k \ne 0) \\ 1 : \text{antiderivative of } \sqrt{x+1} \\ 1 : \text{antiderivative of } 5 - x \\ 1 : \text{evaluation and answer} \end{cases}$

(c) Since $g$ is continuous at $x = 3$, $2k = 3m + 2$.

$$g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}} & \text{for } 0 < x < 3 \\ m & \text{for } 3 < x < 5 \end{cases}$$

$\lim\limits_{x \to 3^-} g'(x) = \frac{k}{4}$ and $\lim\limits_{x \to 3^+} g'(x) = m$

Since these two limits exist and $g$ is differentiable at $x = 3$, the two limits are equal. Thus $\frac{k}{4} = m$.

$8m = 3m + 2;\ m = \frac{2}{5}$ and $k = \frac{8}{5}$

$3 : \begin{cases} 1 : 2k = 3m + 2 \\ 1 : \frac{k}{4} = m \\ 1 : \text{values for } k \text{ and } m \end{cases}$



 95

# Riemann Sums

93 36. If the definite integral $\int_0^2 e^{x^2}\,dx$ is first approximated by using two inscribed rectangles of equal width and then approximated by using the trapezoidal rule with $n=2$, the difference between the two approximations is

(A) 53.60 (B) 30.51 (C) 27.80 (D) 26.80 (E) 12.78

| $x$ | 2 | 5 | 7 | 8 |
|---|---|---|---|---|
| $f(x)$ | 10 | 30 | 40 | 20 |

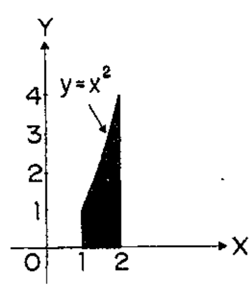
98 85. The function $f$ is continuous on the closed interval $[2,8]$ and has values that are given in the table above. Using the subintervals $[2,5]$, $[5,7]$, and $[7,8]$, what is the trapezoidal approximation of $\int_2^8 f(x)\,dx$?

(A) 110 (B) 130 (C) 160 (D) 190 (E) 210

| $x$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $f(x)$ | 3 | 3 | 5 | 8 | 13 |

97 89. A table of values for a continuous function $f$ is shown above. If four equal subintervals of $[0,2]$ are used, which of the following is the trapezoidal approximation of $\int_0^2 f(x)\,dx$?

~~(A) 8 (B) 12 (C) 16 (D) 24 (E) 32~~




93 42. Calculate the approximate area of the shaded region in the figure by the trapezoidal rule, using divisions at $x=\frac{4}{3}$ and $x=\frac{5}{3}$.

(A) $\frac{50}{27}$ (B) $\frac{251}{108}$ (C) $\frac{7}{3}$ (D) $\frac{127}{54}$ (E) $\frac{77}{27}$

88 18. If three equal subdivisions of $[-4, 2]$ are used, what is the trapezoidal approximation of

BC $\int_{-4}^{2} \frac{e^{-x}}{2}\,dx$?

(A) $e^2 + e^0 + e^{-2}$

(B) $e^4 + e^2 + e^0$

(C) $e^4 + 2e^2 + 2e^0 + e^{-2}$

(D) $\frac{1}{2}\left(e^4 + e^2 + e^0 + e^{-2}\right)$

(E) $\frac{1}{2}\left(e^4 + 2e^2 + 2e^0 + e^{-2}\right)$



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# Riemann Sums and Tables

| $t$ (minutes) | 0 | 2 | 5 | 9 | 10 |
|---|---|---|---|---|---|
| $H(t)$ (degrees Celsius) | 66 | 60 | 52 | 44 | 43 |

2011

2. As a pot of tea cools, the temperature of the tea is modeled by a differentiable function $H$ for $0 \le t \le 10$, where time $t$ is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time $t$ are shown in the table above.

   (a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

   (b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

   (c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

   (d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time $t$ is modeled by a differentiable function $B$ for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

---

| $t$ (hours) | 0 | 2 | 5 | 7 | 8 |
|---|---|---|---|---|---|
| $E(t)$ (hundreds of entries) | 0 | 4 | 13 | 21 | 23 |

10A

2. A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon ($t = 0$) and 8 P.M. ($t = 8$). The number of entries in the box $t$ hours after noon is modeled by a differentiable function $E$ for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times $t$ are shown in the table above.

   (a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

   (b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

   (c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function $P$, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight ($t = 12$)?

   (d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.



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| $x$ | 2 | 3 | 5 | 8 | 13 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 4 | –2 | 3 | 6 |

9-A

5. Let $f$ be a function that is twice differentiable for all real numbers. The table above gives values of $f$ for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\, dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\, dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all $x$ in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of $f$ at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of $f$ on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

| $t$ (hours) | 0 | 1 | 3 | 4 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| $L(t)$ (people) | 120 | 156 | 176 | 126 | 150 | 80 | 0 |

8-A

2. Concert tickets went on sale at noon ($t = 0$) and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time $t$ is modeled by a twice-differentiable function $L$ for $0 \le t \le 9$. Values of $L(t)$ at various times $t$ are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. ($t = 5.5$). Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. ($t = 3$), to the nearest whole number?

| Distance from the river's edge (feet) | 0 | 8 | 14 | 22 | 24 |
|---|---|---|---|---|---|
| Depth of the water (feet) | 0 | 7 | 8 | 2 | 0 |

8-B

3. A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t+10}\right)$ for $0 \le t \le 120$ minutes.

   (a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

   (b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

   (c) The scientist proposes the function $f$, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point $x$ feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

   (d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

| $t$ (minutes) | 0 | 2 | 5 | 7 | 11 | 12 |
|---|---|---|---|---|---|---|
| $r'(t)$ (feet per minute) | 5.7 | 4.0 | 2.0 | 1.2 | 0.6 | 0.5 |

7-A

5. The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function $r$ of time $t$, where $t$ is measured in minutes. For $0 < t < 12$, the graph of $r$ is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius $r$ is given by $V = \frac{4}{3}\pi r^3$.)

   (a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

   (b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

   (c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

   (d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$ ? Give a reason for your answer.

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| Distance $x$ (cm) | 0 | 1 | 5 | 6 | 8 |
|---|---|---|---|---|---|
| Temperature $T(x)$ (°C) | 100 | 93 | 70 | 62 | 55 |

5-A

3. A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire $x$ cm from the heated end. The function $T$ is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every $x$ in the interval $0 < x < 8$? Explain your answer.

| $t$ (minutes) | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|---|---|
| $v(t)$ (miles per minute) | 7.0 | 9.2 | 9.5 | 7.0 | 4.5 | 2.4 | 2.4 | 4.3 | 7.3 |

4-B

3. A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time $t$ minutes, where $v$ is a differentiable function of $t$. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function $f$, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicate units of measure.

(d) According to the model $f$, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?



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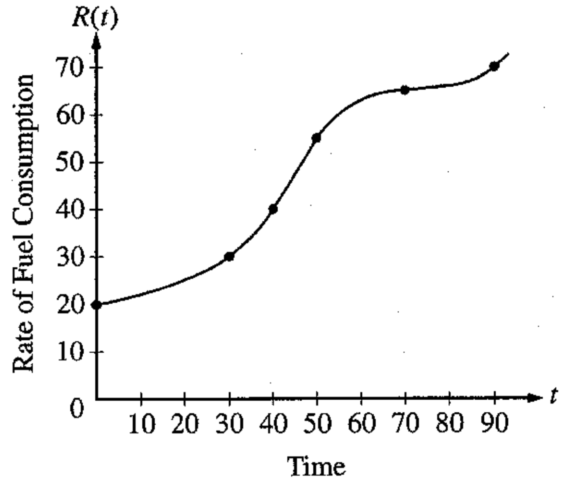


| $t$ (minutes) | $R(t)$ (gallons per minute) |
|---|---|
| 0 | 20 |
| 30 | 30 |
| 40 | 40 |
| 50 | 55 |
| 70 | 65 |
| 90 | 70 |

3-A

3. The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function $R$ of time $t$. The graph of $R$ and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.

   (a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

   (b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$ ? Explain your reasoning.

   (c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$ ? Explain your reasoning.

   (d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

| Distance $x$ (mm) | 0 | 60 | 120 | 180 | 240 | 300 | 360 |
|---|---|---|---|---|---|---|---|
| Diameter $B(x)$ (mm) | 24 | 30 | 28 | 30 | 26 | 24 | 26 |

3-B

3. A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where $x$ represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

   (a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

   (b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

   (c) Using correct units, explain the meaning of $\pi\int_{125}^{275}\left(\frac{B(x)}{2}\right)^2 dx$ in terms of the blood vessel.

   (d) Explain why there must be at least one value $x$, for $0 < x < 360$, such that $B''(x) = 0$.

| $t$ (minutes) | 0 | 4 | 9 | 15 | 20 |
|---|---|---|---|---|---|
| $W(t)$ (degrees Fahrenheit) | 55.0 | 57.1 | 61.8 | 67.9 | 71.0 |

12-A

1. The temperature of water in a tub at time $t$ is modeled by a strictly increasing, twice-differentiable function $W$, where $W(t)$ is measured in degrees Fahrenheit and $t$ is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times $t$ for the first 20 minutes are given in the table above.

   (a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

   (b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

   (c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

   (d) For $20 \le t \le 25$, the function $W$ that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$ ?



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# AP® CALCULUS AB
# 2011 SCORING GUIDELINES

## Question 2

| $t$ (minutes) | 0 | 2 | 5 | 9 | 10 |
|---|---|---|---|---|---|
| $H(t)$ (degrees Celsius) | 66 | 60 | 52 | 44 | 43 |

As a pot of tea cools, the temperature of the tea is modeled by a differentiable function $H$ for $0 \le t \le 10$, where time $t$ is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time $t$ are shown in the table above.

(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

(d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time $t$ is modeled by a differentiable function $B$ for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

---

| Solution | Scoring |
|---|---|
| (a) $H'(3.5) \approx \frac{H(5) - H(2)}{5 - 2} = \frac{52 - 60}{3} = -2.666$ or $-2.667$ degrees Celsius per minute | 1 : answer |
| (b) $\frac{1}{10}\int_0^{10} H(t)\,dt$ is the average temperature of the tea, in degrees Celsius, over the 10 minutes.<br>$\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10}\left(2 \cdot \frac{66+60}{2} + 3 \cdot \frac{60+52}{2} + 4 \cdot \frac{52+44}{2} + 1 \cdot \frac{44+43}{2}\right) = 52.95$ | 3 : { 1 : meaning of expression<br>1 : trapezoidal sum<br>1 : estimate |
| (c) $\int_0^{10} H'(t)\,dt = H(10) - H(0) = 43 - 66 = -23$<br>The temperature of the tea drops 23 degrees Celsius from time $t = 0$ to time $t = 10$ minutes. | 2 : { 1 : value of integral<br>1 : meaning of expression |
| (d) $B(10) = 100 + \int_0^{10} B'(t)\,dt = 34.18275$; $H(10) - B(10) = 8.817$<br>The biscuits are 8.817 degrees Celsius cooler than the tea. | 3 : { 1 : integrand<br>1 : uses $B(0) = 100$<br>1 : answer |

# AP® CALCULUS AB
# 2010 SCORING GUIDELINES

## Question 2

| $t$ (hours) | 0 | 2 | 5 | 7 | 8 |
|---|---|---|---|---|---|
| $E(t)$ (hundreds of entries) | 0 | 4 | 13 | 21 | 23 |

A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box $t$ hours after noon is modeled by a differentiable function $E$ for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times $t$ are shown in the table above.

(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function $P$, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight $(t = 12)$?

(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

---

(a) $E'(6) \approx \dfrac{E(7) - E(5)}{7 - 5} = 4$ hundred entries per hour

1 : answer

(b) $\frac{1}{8}\int_0^8 E(t)\,dt \approx$

$$\frac{1}{8}\left(2\cdot\frac{E(0)+E(2)}{2} + 3\cdot\frac{E(2)+E(5)}{2} + 2\cdot\frac{E(5)+E(7)}{2} + 1\cdot\frac{E(7)+E(8)}{2}\right)$$

$= 10.687$ or $10.688$

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average number of hundreds of entries in the box between noon and 8 P.M.

3 : { 1 : trapezoidal sum; 1 : approximation; 1 : meaning }

(c) $23 - \int_8^{12} P(t)\,dt = 23 - 16 = 7$ hundred entries

2 : { 1 : integral; 1 : answer }

(d) $P'(t) = 0$ when $t = 9.183503$ and $t = 10.816497$.

| $t$ | $P(t)$ |
|---|---|
| 8 | 0 |
| 9.183503 | 5.088662 |
| 10.816497 | 2.911338 |
| 12 | 8 |

Entries are being processed most quickly at time $t = 12$.

3 : { 1 : considers $P'(t) = 0$; 1 : identifies candidates; 1 : answer with justification }




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## Question 5

| $x$ | 2 | 3 | 5 | 8 | 13 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 4 | –2 | 3 | 6 |

Let $f$ be a function that is twice differentiable for all real numbers. The table above gives values of $f$ for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all $x$ in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of $f$ at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of $f$ on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

| Solution | Points |
|---|---|
| (a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$ | 1 : answer |
| (b) $\int_2^{13} (3 - 5f'(x))\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$ $= 3(13 - 2) - 5(f(13) - f(2)) = 8$ | 2 : { 1 : uses Fundamental Theorem of Calculus; 1 : answer } |
| (c) $\int_2^{13} f(x)\,dx \approx f(2)(3 - 2) + f(3)(5 - 3) + f(5)(8 - 5) + f(8)(13 - 8) = 18$ | 2 : { 1 : left Riemann sum; 1 : answer } |
| (d) An equation for the tangent line is $y = -2 + 3(x - 5)$. Since $f''(x) < 0$ for all $x$ in the interval $5 \le x \le 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all $x$ in the interval $5 < x \le 8$. Therefore, $f(7) \le -2 + 3 \cdot 2 = 4$. <br><br> An equation for the secant line is $y = -2 + \frac{5}{3}(x - 5)$. Since $f''(x) < 0$ for all $x$ in the interval $5 \le x \le 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all $x$ in the interval $5 < x < 8$. Therefore, $f(7) \ge -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$. | 4 : { 1 : tangent line; 1 : shows $f(7) \le 4$; 1 : secant line; 1 : shows $f(7) \ge \frac{4}{3}$ } |

## AP® CALCULUS AB
## 2008 SCORING GUIDELINES

### Question 2

| $t$ (hours) | 0 | 1 | 3 | 4 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| $L(t)$ (people) | 120 | 156 | 176 | 126 | 150 | 80 | 0 |

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time $t$ is modeled by a twice-differentiable function $L$ for $0 \le t \le 9$. Values of $L(t)$ at various times $t$ are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

---

(a) $L'(5.5) \approx \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

$2: \begin{cases} 1: \text{estimate} \\ 1: \text{units} \end{cases}$

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4}\left(\frac{L(0) + L(1)}{2}(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3)\right)$$

$= 155.25$ people

$2: \begin{cases} 1: \text{trapezoidal sum} \\ 1: \text{answer} \end{cases}$

(c) $L$ is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some $t$ in $(1, 3)$ and some $t$ in $(4, 7)$. Similarly, $L'(t) < 0$ for some $t$ in $(3, 4)$ and some $t$ in $(7, 8)$. Then, since $L'$ is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of $t$ in $[0, 9]$.

$3: \begin{cases} 1: \text{considers change in sign of } L' \\ 1: \text{analysis} \\ 1: \text{conclusion} \end{cases}$

OR

The continuity of $L$ on $[1, 4]$ implies that $L$ attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, $L$ attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. $L$ is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of $t$ in $[0, 9]$.

OR

$3: \begin{cases} 1: \text{considers relative extrema of } L \text{ on } (0, 9) \\ 1: \text{analysis} \\ 1: \text{conclusion} \end{cases}$

[*Note: There is a function $L$ that satisfies the given conditions with $L'(t) = 0$ for exactly three values of $t$.*]

(d) $\int_0^3 r(t)\,dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

$2: \begin{cases} 1: \text{integrand} \\ 1: \text{limits and answer} \end{cases}$




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## Question 3

| Distance from the river's edge (feet) | 0 | 8 | 14 | 22 | 24 |
|---|---|---|---|---|---|
| Depth of the water (feet) | 0 | 7 | 8 | 2 | 0 |

A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t + 10}\right)$ for $0 \le t \le 120$ minutes.

(a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

(b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

(c) The scientist proposes the function $f$, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point $x$ feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

(d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

---

(a) $\frac{(0+7)}{2}\cdot 8 + \frac{(7+8)}{2}\cdot 6 + \frac{(8+2)}{2}\cdot 8 + \frac{(2+0)}{2}\cdot 2$
$= 115 \text{ ft}^2$

1 : trapezoidal approximation

---

(b) $\frac{1}{120}\int_0^{120} 115v(t)\,dt$
$= 1807.169$ or $1807.170 \text{ ft}^3/\text{min}$

3 : { 1 : limits and average value constant; 1 : integrand; 1 : answer }

(c) $\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right) dx = 122.230$ or $122.231 \text{ ft}^2$

2 : { 1 : integral; 1 : answer }

(d) Let $C$ be the cross-sectional area approximation from part (c). The average volumetric flow is
$\frac{1}{20}\int_{40}^{60} C \cdot v(t)\,dt = 2181.912$ or $2181.913 \text{ ft}^3/\text{min}$.

Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds 2100 $\text{ft}^3/\text{min}$.

3 : { 1 : volumetric flow integral; 1 : average volumetric flow; 1 : answer with reason }




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# AP® CALCULUS AB
# 2007 SCORING GUIDELINES

## Question 5

| $t$ (minutes) | 0 | 2 | 5 | 7 | 11 | 12 |
|---|---|---|---|---|---|---|
| $r'(t)$ (feet per minute) | 5.7 | 4.0 | 2.0 | 1.2 | 0.6 | 0.5 |

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function $r$ of time $t$, where $t$ is measured in minutes. For $0 < t < 12$, the graph of $r$ is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius $r$ is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$ ? Give a reason for your answer.

| | |
|---|---|
| (a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft<br>Since the graph of $r$ is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$. | $2: \begin{cases} 1: \text{estimate} \\ 1: \text{conclusion with reason} \end{cases}$ |
| (b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$<br>$\left.\frac{dV}{dt}\right\|_{t=5} = 4\pi(30)^2 2 = 7200\pi \text{ ft}^3/\text{min}$ | $3: \begin{cases} 2: \frac{dV}{dt} \\ 1: \text{answer} \end{cases}$ |
| (c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$<br>$= 19.3$ ft<br>$\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes. | $2: \begin{cases} 1: \text{approximation} \\ 1: \text{explanation} \end{cases}$ |
| (d) Since $r$ is concave down, $r'$ is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$. | 1 : conclusion with reason |
| Units of $\text{ft}^3/\text{min}$ in part (b) and ft in part (c) | 1 : units in (b) and (c) |

## Question 3

| Distance $x$ (cm) | 0 | 1 | 5 | 6 | 8 |
|---|---|---|---|---|---|
| Temperature $T(x)$ (°C) | 100 | 93 | 70 | 62 | 55 |

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire $x$ cm from the heated end. The function $T$ is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every $x$ in the interval $0 < x < 8$ ? Explain your answer.

---

(a) $\dfrac{T(8) - T(6)}{8 - 6} = \dfrac{55 - 62}{2} = -\dfrac{7}{2}\,°\text{C/cm}$

1 : answer

(b) $\dfrac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$$A = \frac{100 + 93}{2}\cdot 1 + \frac{93 + 70}{2}\cdot 4 + \frac{70 + 62}{2}\cdot 1 + \frac{62 + 55}{2}\cdot 2$$

Average temperature $\approx \dfrac{1}{8}A = 75.6875\,°\text{C}$

3 : { 1 : $\dfrac{1}{8}\int_0^8 T(x)\,dx$; 1 : trapezoidal sum; 1 : answer }

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45\,°\text{C}$

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

2 : { 1 : value; 1 : meaning }

(d) Average rate of change of temperature on $[1, 5]$ is $\dfrac{70 - 93}{5 - 1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\dfrac{62 - 70}{6 - 5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some $c_1$ in the interval $(1, 5)$ and $T'(c_2) = -8$ for some $c_2$ in the interval $(5, 6)$. It follows that $T'$ must decrease somewhere in the interval $(c_1, c_2)$. Therefore $T''$ is not positive for every $x$ in $[0, 8]$.

2 : { 1 : two slopes of secant lines; 1 : answer with explanation }

Units of °C/cm in (a), and °C in (b) and (c)

1 : units in (a), (b), and (c)

### Question 3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time $t$ minutes, where $v$ is a differentiable function of $t$. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

| $t$ (min) | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|---|---|
| $v(t)$ (mpm) | 7.0 | 9.2 | 9.5 | 7.0 | 4.5 | 2.4 | 2.4 | 4.3 | 7.3 |

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function $f$, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$ ? Indicates units of measure.

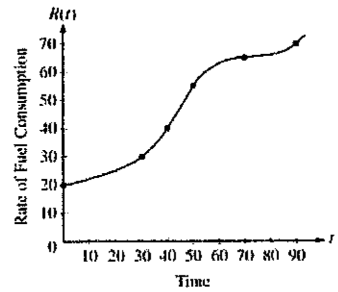
(d) According to the model $f$, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

---

| | |
|---|---|
| (a) Midpoint Riemann sum is $10\cdot[v(5) + v(15) + v(25) + v(35)]$ $= 10\cdot[9.2 + 7.0 + 2.4 + 4.3] = 229$ The integral gives the total distance in miles that the plane flies during the 40 minutes. | 3 : { 1 : $v(5) + v(15) + v(25) + v(35)$; 1 : answer; 1 : meaning with units } |
| (b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of $t$. | 2 : { 1 : two instances; 1 : justification } |
| (c) $f'(23) = -0.407$ or $-0.408$ miles per minute$^2$ | 1 : answer with units |
| (d) Average velocity $= \frac{1}{40}\int_0^{40} f(t)\,dt$ $= 5.916$ miles per minute | 3 : { 1 : limits; 1 : integrand; 1 : answer } |

### Question 3

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function $R$ of time $t$. The graph of $R$ and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.




| $t$ (minutes) | $R(t)$ (gallons per minute) |
|---|---|
| 0 | 20 |
| 30 | 30 |
| 40 | 40 |
| 50 | 55 |
| 70 | 65 |
| 90 | 70 |

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$ ? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

---

(a) $R'(45) \approx \dfrac{R(50) - R(40)}{50 - 40} = \dfrac{55 - 40}{10}$
$= 1.5 \text{ gal/min}^2$

2 : { 1 : a difference quotient using numbers from table and interval that contains 45
1 : 1.5 gal/min$^2$ }

(b) $R''(45) = 0$ since $R'(t)$ has a maximum at $t = 45$.

2 : { 1 : $R''(45) = 0$
1 : reason }

(c) $\int_0^{90} R(t)\,dt \approx (30)(20) + (10)(30) + (10)(40)$
$+ (20)(55) + (20)(65) = 3700$

Yes, this approximation is less because the graph of $R$ is increasing on the interval.

2 : { 1 : value of left Riemann sum
1 : "less" with reason }

(d) $\int_0^b R(t)\,dt$ is the total amount of fuel in gallons consumed for the first $b$ minutes.

$\frac{1}{b}\int_0^b R(t)\,dt$ is the average value of the rate of fuel consumption in gallons/min during the first $b$ minutes.

3 : { 2 : meanings
1 : meaning of $\int_0^b R(t)\,dt$
1 : meaning of $\frac{1}{b}\int_0^b R(t)\,dt$
< −1 > if no reference to time $b$
1 : units in both answers }

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# AP® CALCULUS AB
# 2003 SCORING GUIDELINES (Form B)

## Question 3

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where $x$ represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

| Distance $x$ (mm) | 0 | 60 | 120 | 180 | 240 | 300 | 360 |
|---|---|---|---|---|---|---|---|
| Diameter $B(x)$ (mm) | 24 | 30 | 28 | 30 | 26 | 24 | 26 |

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi \int_{125}^{275} \left( \frac{B(x)}{2} \right)^2 dx$ in terms of the blood vessel.

(d) Explain why there must be at least one value $x$, for $0 < x < 360$, such that $B''(x) = 0$.

---

| | |
|---|---|
| (a) $\frac{1}{360} \int_0^{360} \frac{B(x)}{2} dx$ | 2 : { 1 : limits and constant; 1 : integrand } |
| (b) $\frac{1}{360}\left[120\left(\frac{B(60)}{2} + \frac{B(180)}{2} + \frac{B(300)}{2}\right)\right] =$ $\frac{1}{360}[60(30 + 30 + 24)] = 14$ | 2 : { 1 : $B(60) + B(180) + B(300)$; 1 : answer } |
| (c) $\frac{B(x)}{2}$ is the radius, so $\pi\left(\frac{B(x)}{2}\right)^2$ is the area of the cross section at $x$. The expression is the volume in $mm^3$ of the blood vessel between 125 mm and 275 mm from the end of the vessel. | 2 : { 1 : volume in $mm^3$; 1 : between $x = 125$ and $x = 275$ } |
| (d) By the MVT, $B'(c_1) = 0$ for some $c_1$ in $(60, 180)$ and $B'(c_2) = 0$ for some $c_2$ in $(240, 360)$. The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some $x$ in the interval $(c_1, c_2)$. | 3 : { 2 : explains why there are two values of $x$ where $B'(x)$ has the same value; 1 : explains why that means $B''(x) = 0$ for $0 < x < 360$ } Note: max 1/3 if only explains why $B'(x) = 0$ at some $x$ in $(0, 360)$. |





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