# Motion - AP Multiple Choice

98 16. A particle moves along the $x$-axis so that at any time $t \geq 0$ its position is given by $x(t) = t^3 - 3t^2 - 9t + 1$. For what values of $t$ is the particle at rest?

(A) No values (B) 1 only (C) 3 only (D) 5 only (E) 1 and 3

85 28. If the position of a particle on the $x$-axis at time $t$ is $-5t^2$, then the average velocity of the particle for $0 \leq t \leq 3$ is

(A) $-45$ (B) $-30$ (C) $-15$ (D) $-10$ (E) $-5$

85 11. The position of a particle moving along a straight line at any time $t$ is given by $s(t) = t^2 + 4t + 4$. What is the acceleration of the particle when $t = 4$?

(A) 0 (B) 2 (C) 4 (D) 8 (E) 12

14. A particle moves along the $x$-axis so that its position at time $t$ is given by $x(t) = t^2 - 6t + 5$. For what value of $t$ is the velocity of the particle zero?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

85 BC 25. A particle moves along the $x$-axis so that at any time $t$ its position is given by $x(t) = te^{-2t}$. For what values of $t$ is the particle at rest?

(A) No values (B) 0 only (C) $\frac{1}{2}$ only (D) 1 only (E) 0 and $\frac{1}{2}$

98 24. The maximum acceleration attained on the interval $0 \leq t \leq 3$ by the particle whose velocity is given by $v(t) = t^3 - 3t^2 + 12t + 4$ is

(A) 9 (B) 12 (C) 14 (D) 21 (E) 40

93 26. A particle moves along a line so that at time $t$, where $0 \le t \le \pi$, its position is given by $s(t) = -4\cos t - \frac{t^2}{2} + 10$. What is the velocity of the particle when its acceleration is zero?

(A) $-5.19$ (B) $0.74$ (C) $1.32$ (D) $2.55$ (E) $8.13$

64 AB 19. A point moves on the $x$-axis in such a way that its velocity at time $t$ $(t > 0)$ is given by $v = \frac{\ln t}{t}$. At what value of $t$ does $v$ attain its maximum?

(A) $1$ (B) $e^{\frac{1}{2}}$ (C) $e$ (D) $e^{\frac{3}{2}}$

(E) There is no maximum value for $v$.

93 BC 12. The position of a particle moving along the $x$-axis is $x(t) = \sin(2t) - \cos(3t)$ for time $t \ge 0$. When $t = \pi$, the acceleration of the particle is

(A) $9$ (B) $\frac{1}{9}$ (C) $0$ (D) $-\frac{1}{9}$ (E) $-9$

97 BC 79. The position of an object attached to a spring is given by $y(t) = \frac{1}{6}\cos(5t) - \frac{1}{4}\sin(5t)$, where $t$ is ~~time in seconds. In the first 4 seconds, how many times is the velocity of the object equal to 0?~~

(A) Zero
(B) Three
(C) Five
(D) Six
(E) Seven

69 35. At $t = 0$ a particle starts at rest and moves along a line in such a way that at time $t$ its acceleration is $24t^2$ feet per second per second. Through how many feet does the particle move during the first 2 seconds?

(A) 32 (B) 48 (C) 64 (D) 96 (E) 192

88 3. A particle with velocity at any time $t$ given by $v(t) = e^t$ moves in a straight line. How far does the particle move from $t = 0$ to $t = 2$?

(A) $e^2 - 1$ (B) $e - 1$ (C) $2e$ (D) $e^2$ (E) $\frac{e^3}{3}$

85 BC 15. If the velocity of a particle moving along the $x$-axis is $v(t) = 2t - 4$ and if at $t = 0$ its position is 4, then at any time $t$ its position $x(t)$ is

(A) $t^2 - 4t$ (B) $t^2 - 4t - 4$ (C) $t^2 - 4t + 4$ (D) $2t^2 - 4t$ (E) $2t^2 - 4t + 4$

73 8. A particle moves in a straight line with velocity $v(t) = t^2$. How far does the particle move between times $t = 1$ and $t = 2$?

(A) $\frac{1}{3}$ (B) $\frac{7}{3}$ (C) 3 (D) 7 (E) 8

73 28. A point moves in a straight line so that its distance at time $t$ from a fixed point of the line is $8t - 3t^2$. What is the *total* distance covered by the point between $t = 1$ and $t = 2$?

(A) 1 (B) $\frac{4}{3}$ (C) $\frac{5}{3}$ (D) 2 (E) 5

85 14. The velocity of a particle moving on a line at time $t$ is $v = 3t^{\frac{1}{2}} + 5t^{\frac{3}{2}}$ meters per second. How many meters did the particle travel from $t = 0$ to $t = 4$?

(A) 32 (B) 40 (C) 64 (D) 80 (E) 184

93 11. The acceleration of a particle moving along the $x$-axis at time $t$ is given by $a(t) = 6t - 2$. If the velocity is 25 when $t = 3$ and the position is 10 when $t = 1$, then the position $x(t) =$

(A) $9t^2 + 1$

(B) $3t^2 - 2t + 4$

(C) $t^3 - t^2 + 4t + 6$

(D) $t^3 - t^2 + 9t - 20$

(E) $36t^3 - 4t^2 - 77t + 55$

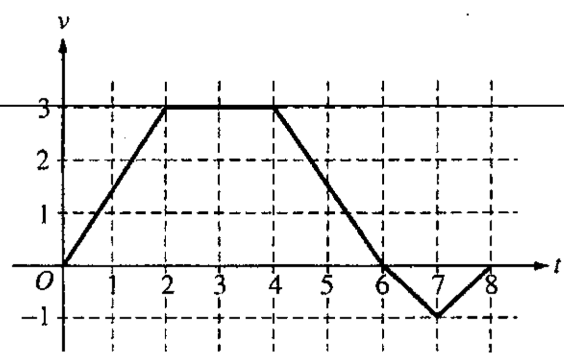
97 BC 13. A particle moves along the $x$-axis so that its acceleration at any time $t$ is $a(t) = 2t - 7$. If the initial velocity of the particle is 6, at what time $t$ during the interval $0 \le t \le 4$ is the particle farthest to the right?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

97 87. At time $t \ge 0$, the acceleration of a particle moving on the $x$-axis is $a(t) = t + \sin t$. At $t = 0$, the velocity of the particle is –2. For what value $t$ will the velocity of the particle be zero?

(A) 1.02 (B) 1.48 (C) 1.85 (D) 2.81 (E) 3.14

Questions 8-9 refer to the following situation.




A bug begins to crawl up a vertical wire at time $t = 0$. The velocity $v$ of the bug at time $t$, $0 \le t \le 8$, is given by the function whose graph is shown above.

97 8. At what value of $t$ does the bug change direction?

(A) 2 (B) 4 (C) 6 (D) 7 (E) 8

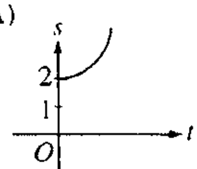
97 9. What is the total distance the bug traveled from $t = 0$ to $t = 8$?

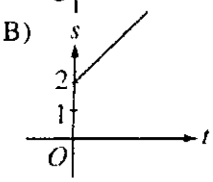
(A) 14 (B) 13 (C) 11 (D) 8 (E) 6

211 211

98 BC

90. A particle starts from rest at the point $(2,0)$ and moves along the $x$-axis with a constant positive acceleration for time $t \geq 0$. Which of the following could be the graph of the distance $s(t)$ of the particle from the origin as a function of time $t$?

(A)


(B)


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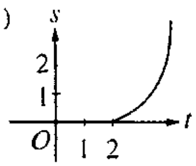
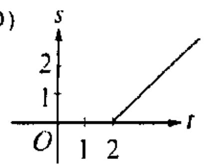
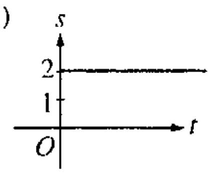


(D)


E)


93 BC

20. A particle moves along the $x$-axis so that at any time $t \geq 0$ the acceleration of the particle is $a(t) = e^{-2t}$. If at $t = 0$ the velocity of the particle is $\frac{5}{2}$ and its position is $\frac{17}{4}$, then its position at any time $t > 0$ is $x(t) =$

(A) $-\dfrac{e^{-2t}}{2} + 3$

(B) $\dfrac{e^{-2t}}{4} + 4$

(C) $4e^{-2t} + \dfrac{9}{2}t + \dfrac{1}{4}$

(D) $\dfrac{e^{-2t}}{2} + 3t + \dfrac{15}{4}$

(E) $\dfrac{e^{-2t}}{4} + 3t + 4$

| $t$ (sec) | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $a(t)$ (ft/sec$^2$) | 5 | 2 | 8 | 3 |

8 C 91. The data for the acceleration $a(t)$ of a car from 0 to 6 seconds are given in the table above. If the velocity at $t=0$ is 11 feet per second, the approximate value of the velocity at $t=6$, computed using a left-hand Riemann sum with three subintervals of equal length, is

(A) 26 ft/sec (B) 30 ft/sec (C) 37 ft/sec (D) 39 ft/sec (E) 41 ft/sec

98 BC 10. A particle moves on a plane curve so that at any time $t>0$ its $x$-coordinate is $t^3-t$ and its $y$-coordinate is $(2t-1)^3$. The acceleration vector of the particle at $t=1$ is

(A) $(0,1)$ (B) $(2,3)$ (C) $(2,6)$ (D) $(6,12)$ (E) $(6,24)$

93 BC 4. A particle moves along the curve $xy=10$. If $x=2$ and $\frac{dy}{dt}=3$, what is the value of $\frac{dx}{dt}$?

(A) $-\frac{5}{2}$ (B) $-\frac{6}{5}$ (C) 0 (D) $\frac{4}{5}$ (E) $\frac{6}{5}$

73 BC 22. A particle moves on the curve $y=\ln x$ so that the $x$-component has velocity $x'(t)=t+1$ for $t\ge 0$. At time $t=0$, the particle is at the point $(1,0)$. At time $t=1$, the particle is at the point

(A) $(2,\ln 2)$ (B) $(e^2,2)$ (C) $\left(\frac{5}{2},\ln\frac{5}{2}\right)$

(D) $(3,\ln 3)$ (E) $\left(\frac{3}{2},\ln\frac{3}{2}\right)$

69 BC 35. At $t=0$ a particle starts at rest and moves along a line in such a way that at time $t$ its acceleration is $24t^2$ feet per second per second. Through how many feet does the particle move during the first 2 seconds?

(A) 32 (B) 48 (C) 64 (D) 96 (E) 192

88 BC 12. A particle travels in a straight line with a constant acceleration of 3 meters per second per second. If the velocity of the particle is 10 meters per second at time 2 seconds, how far does the particle travel during the time interval when its velocity increases from 4 meters per second to 10 meters per second?

(A) 20 m (B) 14 m (C) 7 m (D) 6 m (E) 3 m

69 BC 19. A point moves on the $x$-axis in such a way that its velocity at time $t$ $(t > 0)$ is given by $v = \frac{\ln t}{t}$.

At what value of $t$ does $v$ attain its maximum?

(A) 1 (B) $e^{\frac{1}{2}}$ (C) $e$ (D) $e^{\frac{3}{2}}$

(E) There is no maximum value for $v$.



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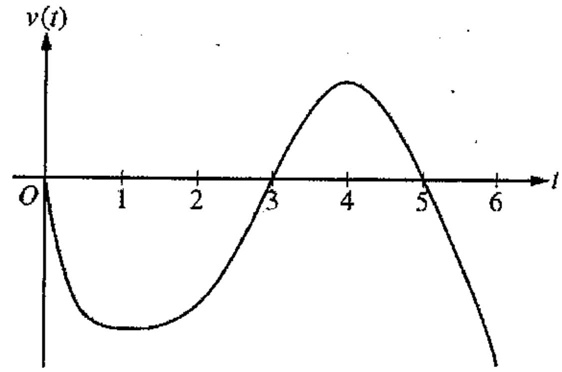
# Motion - AP FR Questions




Graph of $v$

8-A

4. A particle moves along the $x$-axis so that its velocity at time $t$, for $0 \le t \le 6$, is given by a differentiable function $v$ whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the $t$-axis and the graph of $v$ on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

   (a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

   (b) For how many values of $t$, where $0 \le t \le 6$, is the particle at $x = -8$ ? Explain your reasoning.

   (c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

   (d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

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3-B

4. A particle moves along the $x$-axis with velocity at time $t \ge 0$ given by $v(t) = -1 + e^{1-t}$.

   (a) Find the acceleration of the particle at time $t = 3$.

   (b) Is the speed of the particle increasing at time $t = 3$ ? Give a reason for your answer.

   (c) Find all values of $t$ at which the particle changes direction. Justify your answer.

   (d) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.

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1. For $0 \le t \le 6$, a particle is moving along the $x$-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

   (a) Is the speed of the particle increasing or decreasing at time $t = 5.5$ ? Give a reason for your answer.

   (b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

   (c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

   (d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

10-B

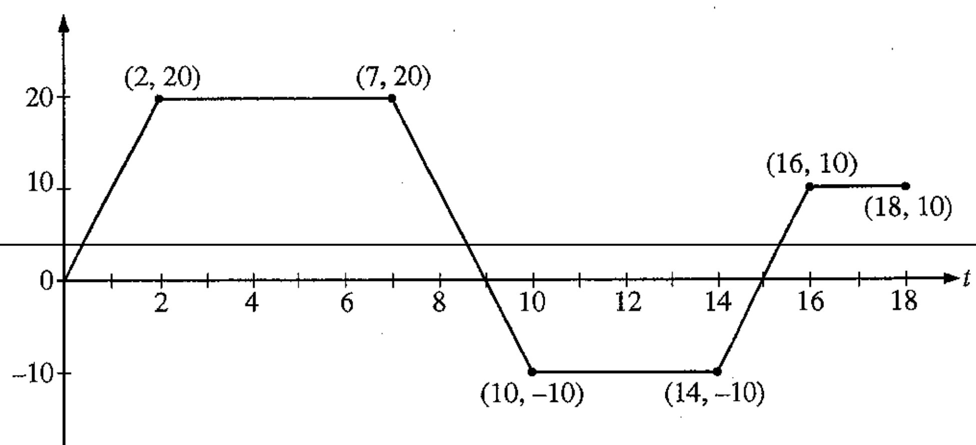
6. Two particles move along the $x$-axis. For $0 \le t \le 6$, the position of particle $P$ at time $t$ is given by $p(t) = 2\cos\left(\frac{\pi}{4}t\right)$, while the position of particle $R$ at time $t$ is given by $r(t) = t^3 - 6t^2 + 9t + 3$.

   (a) For $0 \le t \le 6$, find all times $t$ during which particle $R$ is moving to the right.

   (b) For $0 \le t \le 6$, find all times $t$ during which the two particles travel in opposite directions.

   (c) Find the acceleration of particle $P$ at time $t = 3$. Is particle $P$ speeding up, slowing down, or doing neither at time $t = 3$ ? Explain your reasoning.

   (d) Write, but do not evaluate, an expression for the average distance between the two particles on the interval $1 \le t \le 3$.




Graph of $v$

10-B

4. A squirrel starts at building $A$ at time $t = 0$ and travels along a straight, horizontal wire connected to building $B$. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.

   (a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

   (b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building $A$ ? How far from building $A$ is the squirrel at that time?

   (c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

   (d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building $A$ that are valid for the time interval $7 < t < 10$.

**5-B**

3. A particle moves along the $x$-axis so that its velocity $v$ at time $t$, for $0 \le t \le 5$, is given by $v(t) = \ln\left(t^2 - 3t + 3\right)$. The particle is at position $x = 8$ at time $t = 0$.

   (a) Find the acceleration of the particle at time $t = 4$.

   (b) Find all times $t$ in the open interval $0 < t < 5$ at which the particle changes direction. During which time intervals, for $0 \le t \le 5$, does the particle travel to the left?

   (c) Find the position of the particle at time $t = 2$.

   (d) Find the average speed of the particle over the interval $0 \le t \le 2$.

**3-A**

2. A particle moves along the $x$-axis so that its velocity at time $t$ is given by

$$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right).$$

   At time $t = 0$, the particle is at position $x = 1$.

   (a) Find the acceleration of the particle at time $t = 2$. Is the speed of the particle increasing at $t = 2$? Why or why not?

   (b) Find all times $t$ in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.

   (c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

   (d) During the time interval $0 \le t \le 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

**2-A-**

3. An object moves along the $x$-axis with initial position $x(0) = 2$. The velocity of the object at time $t \ge 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

   (a) What is the acceleration of the object at time $t = 4$?

   (b) Consider the following two statements.

      Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.
      Statement II: For $3 < t < 4.5$, the speed of the object is increasing.

      Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

   (c) What is the total distance traveled by the object over the time interval $0 \le t \le 4$?

   (d) What is the position of the object at time $t = 4$?



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| $t$ (sec) | 0 | 15 | 25 | 30 | 35 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $v(t)$ (ft/sec) | −20 | −30 | −20 | −14 | −10 | 0 | 10 |
| $a(t)$ $(\text{ft}/\text{sec}^2)$ | 1 | 5 | 2 | 1 | 2 | 4 | 2 |

6-B

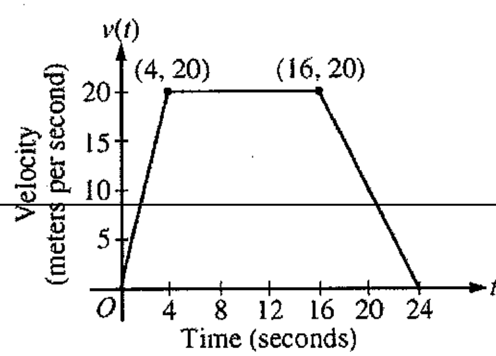
6. A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity $v$, measured in feet per second, and acceleration $a$, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

   (a) Using appropriate units, explain the meaning of $\int_{30}^{60} |v(t)|\, dt$ in terms of the car's motion. Approximate $\int_{30}^{60} |v(t)|\, dt$ using a trapezoidal approximation with the three subintervals determined by the table.

   (b) Using appropriate units, explain the meaning of $\int_{0}^{30} a(t)\, dt$ in terms of the car's motion. Find the exact value of $\int_{0}^{30} a(t)\, dt$.

   (c) For $0 < t < 60$, must there be a time $t$ when $v(t) = -5$ ? Justify your answer.

   (d) For $0 < t < 60$, must there be a time $t$ when $a(t) = 0$ ? Justify your answer.




5-A

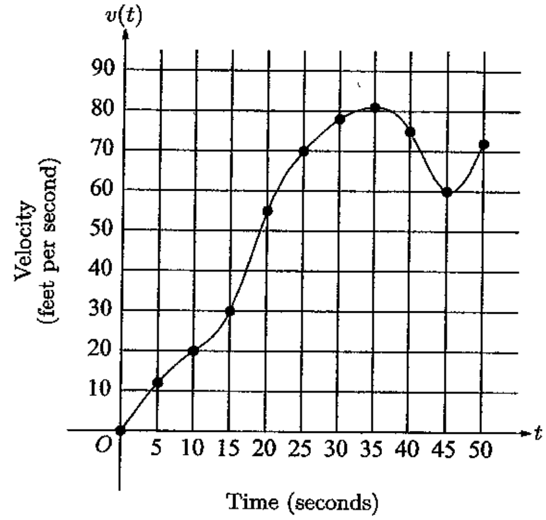
5. A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

   (a) Find $\int_{0}^{24} v(t)\, dt$. Using correct units, explain the meaning of $\int_{0}^{24} v(t)\, dt$.

   (b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

   (c) Let $a(t)$ be the car's acceleration at time $t$, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

   (d) Find the average rate of change of $v$ over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of $c$, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?




| $t$ (seconds) | $v(t)$ (feet per second) |
|---|---|
| 0 | 0 |
| 5 | 12 |
| 10 | 20 |
| 15 | 30 |
| 20 | 55 |
| 25 | 70 |
| 30 | 78 |
| 35 | 81 |
| 40 | 75 |
| 45 | 60 |
| 50 | 72 |

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3. The graph of the velocity $v(t)$, in ft/sec, of a car traveling on a straight road, for $0 \le t \le 50$, is shown above. A table of values for $v(t)$, at 5 second intervals of time $t$, is shown to the right of the graph.
   (a) During what intervals of time is the acceleration of the car positive? Give a reason for your answer.
   (b) Find the average acceleration of the car, in ft/sec$^2$, over the interval $0 \le t \le 50$.
   (c) Find one approximation for the acceleration of the car, in ft/sec$^2$, at $t = 40$. Show the computations you used to arrive at your answer.
   (d) Approximate $\int_0^{50} v(t)\,dt$ with a Riemann sum, using the midpoints of five subintervals of equal length. Using correct units, explain the meaning of this integral.

| $t$ (seconds) | 0 | 10 | 40 | 60 |
|---|---|---|---|---|
| $B(t)$ (meters) | 100 | 136 | 9 | 49 |
| $v(t)$ (meters per second) | 2.0 | 2.3 | 2.5 | 4.6 |

118 5. Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function $B$ models Ben's position on the track, measured in meters from the western end of the track, at time $t$, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times $t$.
   (a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.
   (b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\,dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\,dt$ using a left Riemann sum with the subintervals indicated by the data in the table.
   (c) For $40 \le t \le 60$, must there be a time $t$ when Ben's velocity is 2 meters per second? Justify your answer.
   (d) A light is directly above the western end of the track. Ben rides so that at time $t$, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$ ?

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| $t$ (seconds) | 0 | 8 | 20 | 25 | 32 | 40 |
|---|---|---|---|---|---|---|
| $v(t)$ (meters per second) | 3 | 5 | −10 | −8 | −4 | 7 |

9-B

6. The velocity of a particle moving along the $x$-axis is modeled by a differentiable function $v$, where the position $x$ is measured in meters, and time $t$ is measured in seconds. Selected values of $v(t)$ are given in the table above. The particle is at position $x = 7$ meters when $t = 0$ seconds.

   (a) Estimate the acceleration of the particle at $t = 36$ seconds. Show the computations that lead to your answer. Indicate units of measure.

   (b) Using correct units, explain the meaning of $\int_{20}^{40} v(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate $\int_{20}^{40} v(t)\,dt$.

   (c) For $0 \le t \le 40$, must the particle change direction in any of the subintervals indicated by the data in the table? If so, identify the subintervals and explain your reasoning. If not, explain why not.

   (d) Suppose that the acceleration of the particle is positive for $0 < t < 8$ seconds. Explain why the position of the particle at $t = 8$ seconds must be greater than $x = 30$ meters.

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| $t$ (seconds) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|---|
| $v(t)$ (feet per second) | 5 | 14 | 22 | 29 | 35 | 40 | 44 | 47 | 49 |

6-A

4. Rocket $A$ has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of $t$ over the interval $0 \le t \le 80$ seconds, as shown in the table above.

   (a) Find the average acceleration of rocket $A$ over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

   (b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

   (c) Rocket $B$ is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.



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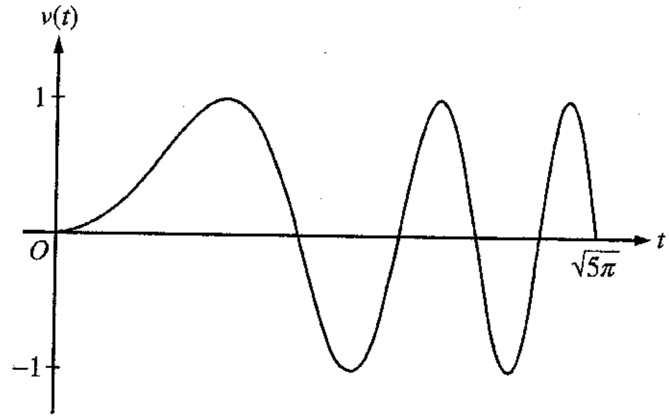



7-B

2. A particle moves along the $x$-axis so that its velocity $v$ at time $t \geq 0$ is given by $v(t) = \sin\left(t^2\right)$. The graph of $v$ is shown above for $0 \leq t \leq \sqrt{5\pi}$. The position of the particle at time $t$ is $x(t)$ and its position at time $t = 0$ is $x(0) = 5$.

   (a) Find the acceleration of the particle at time $t = 3$.

   (b) Find the total distance traveled by the particle from time $t = 0$ to $t = 3$.

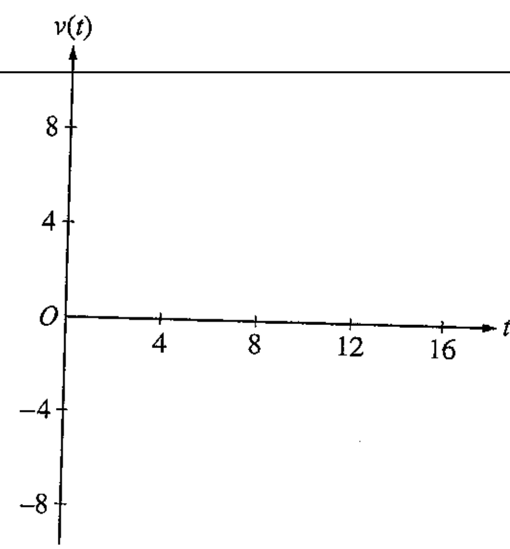
   (c) Find the position of the particle at time $t = 3$.

   (d) For $0 \leq t \leq \sqrt{5\pi}$, find the time $t$ at which the particle is farthest to the right. Explain your answer.

2-B

3. A particle moves along the $x$-axis so that its velocity $v$ at any time $t$, for $0 \leq t \leq 16$, is given by $v(t) = e^{2\sin t} - 1$. At time $t = 0$, the particle is at the origin.

   (a) On the axes provided, sketch the graph of $v(t)$ for $0 \leq t \leq 16$.
   **(Note: Use the axes provided in the test booklet.)**




   (b) During what intervals of time is the particle moving to the left? Give a reason for your answer.

   (c) Find the total distance traveled by the particle from $t = 0$ to $t = 4$.

   (d) Is there any time $t$, $0 < t \leq 16$, at which the particle returns to the origin? Justify your answer.

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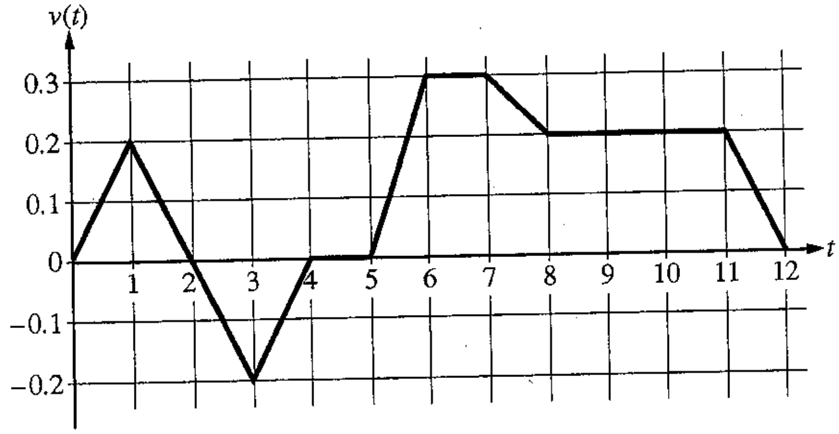



09-A

1. Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

   (a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

   (b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

   (c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

   (d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function $w$ given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

---

4-A

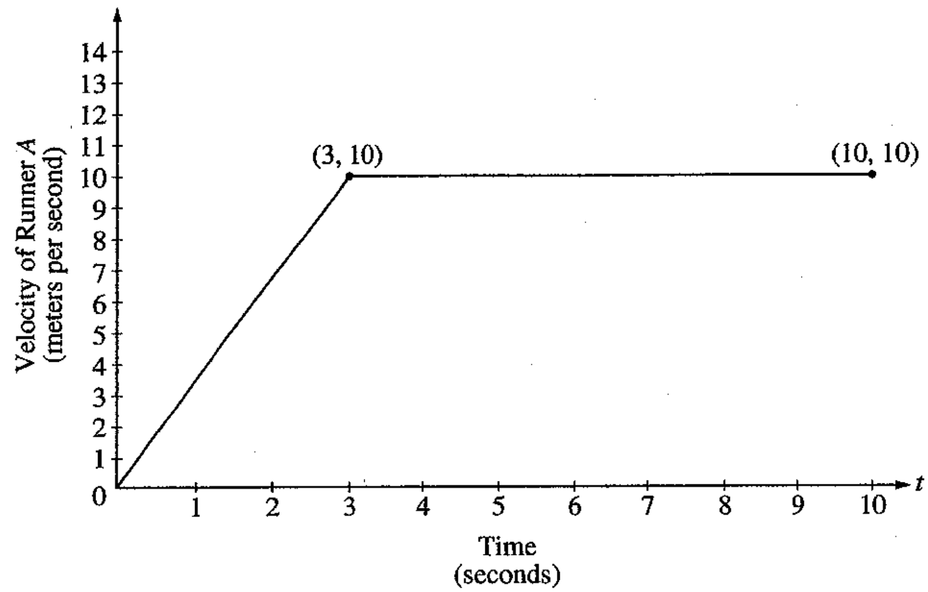
3. A particle moves along the $y$-axis so that its velocity $v$ at time $t \ge 0$ is given by $v(t) = 1 - \tan^{-1}\left(e^t\right)$. At time $t = 0$, the particle is at $y = -1$. (Note: $\tan^{-1} x = \arctan x$)

   (a) Find the acceleration of the particle at time $t = 2$.

   (b) Is the speed of the particle increasing or decreasing at time $t = 2$ ? Give a reason for your answer.

   (c) Find the time $t \ge 0$ at which the particle reaches its highest point. Justify your answer.

   (d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$ ? Justify your answer.

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2. Two runners, $A$ and $B$, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner $A$. The velocity, in meters per second, of Runner $B$ is given by the function $v$ defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner $A$ and the velocity of Runner $B$ at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner $A$ and the acceleration of Runner $B$ at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner $A$ and the total distance run by Runner $B$ over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

7A 4. A particle moves along the $x$-axis with position at time $t$ given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$.

(a) Find the time $t$ at which the particle is farthest to the left. Justify your answer.

~~(b) Find the value of the constant $A$ for which $x(t)$ satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$~~ for $0 < t < 2\pi$.

12A 6. For $0 \le t \le 12$, a particle moves along the $x$-axis. The velocity of the particle at time $t$ is given by $v(t) = \cos\left(\dfrac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$.

(a) For $0 \le t \le 12$, when is the particle moving to the left?

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

(c) Find the acceleration of the particle at time $t$. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

(d) Find the position of the particle at time $t = 4$.

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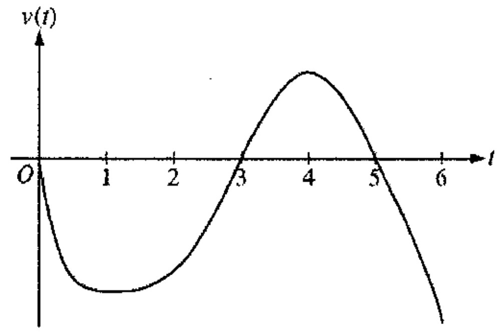
### Question 4




Graph of $v$

A particle moves along the $x$-axis so that its velocity at time $t$, for $0 \le t \le 6$, is given by a differentiable function $v$ whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the $t$-axis and the graph of $v$ on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of $t$, where $0 \le t \le 6$, is the particle at $x = -8$ ? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

| | |
|---|---|
| (a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$.<br>$x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$<br>$x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$<br>Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$. | $3: \begin{cases} 1: \text{identifies } t = 3 \text{ as a candidate} \\ 1: \text{considers } \int_0^6 v(t)\,dt \\ 1: \text{conclusion} \end{cases}$ |
| (b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$.<br>By the Intermediate Value Theorem, there are three values of $t$ for which the particle is at $x(t) = -8$. | $3: \begin{cases} 1: \text{positions at } t = 3,\ t = 5, \\ \quad \text{and } t = 6 \\ 1: \text{description of motion} \\ 1: \text{conclusion} \end{cases}$ |
| (c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and $v$ is increasing. | 1 : answer with reason |
| (d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals. | $2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$ |



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## Question 4

A particle moves along the $x$-axis with velocity at time $t \geq 0$ given by $v(t) = -1 + e^{1-t}$.

(a) Find the acceleration of the particle at time $t = 3$.

(b) Is the speed of the particle increasing at time $t = 3$? Give a reason for your answer.

(c) Find all values of $t$ at which the particle changes direction. Justify your answer.

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

---

(a) $a(t) = v'(t) = -e^{1-t}$

$a(3) = -e^{-2}$

$2 : \begin{cases} 1 : v'(t) \\ 1 : a(3) \end{cases}$

(b) $a(3) < 0$

$v(3) = -1 + e^{-2} < 0$

Speed is increasing since $v(3) < 0$ and $a(3) < 0$.

1 : answer with reason

(c) $v(t) = 0$ when $1 = e^{1-t}$, so $t = 1$.

$v(t) > 0$ for $t < 1$ and $v(t) < 0$ for $t > 1$.

Therefore, the particle changes direction at $t = 1$.

$2 : \begin{cases} 1 : \text{solves } v(t) = 0 \text{ to get } t = 1 \\ 1 : \text{justifies change in direction at } t = 1 \end{cases}$

(d)
$$\begin{aligned} \text{Distance} &= \int_0^3 |v(t)|\,dt \\ &= \int_0^1 \left(-1 + e^{1-t}\right) dt + \int_1^3 \left(1 - e^{1-t}\right) dt \\ &= \left(-t - e^{1-t}\Big|_0^1\right) + \left(t + e^{1-t}\Big|_1^3\right) \\ &= (-1 - 1 + e) + \left(3 + e^{-2} - 1 - 1\right) \\ &= e + e^{-2} - 1 \end{aligned}$$

$4 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{antidifferentiation} \\ 1 : \text{evaluation} \end{cases}$

OR

$x(t) = -t - e^{1-t}$

$x(0) = -e$

$x(1) = -2$

$x(3) = -e^{-2} - 3$

$$\begin{aligned} \text{Distance} &= (x(1) - x(0)) + (x(1) - x(3)) \\ &= (-2 + e) + \left(1 + e^{-2}\right) \\ &= e + e^{-2} - 1 \end{aligned}$$

OR

$4 : \begin{cases} 1 : \text{any antiderivative} \\ 1 : \text{evaluates } x(t) \text{ when } t = 0, 1, 3 \\ 1 : \text{evaluates distance between points} \\ 1 : \text{evaluates total distance} \end{cases}$





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# AP® CALCULUS AB
# 2011 SCORING GUIDELINES

## Question 1

For $0 \le t \le 6$, a particle is moving along the $x$-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

(a) Is the speed of the particle increasing or decreasing at time $t = 5.5$ ? Give a reason for your answer.

(b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

(c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

(d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

| | |
|---|---|
| (a) $v(5.5) = -0.45337,\ a(5.5) = -1.35851$<br><br>The speed is increasing at time $t = 5.5$, because velocity and acceleration have the same sign. | 2 : conclusion with reason |
| (b) Average velocity $= \frac{1}{6}\int_0^6 v(t)\,dt = 1.949$ | $2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ |
| (c) Distance $= \int_0^6 \lvert v(t) \rvert\,dt = 12.573$ | $2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ |
| (d) $v(t) = 0$ when $t = 5.19552$. Let $b = 5.19552$.<br>$v(t)$ changes sign from positive to negative at time $t = b$.<br>$x(b) = 2 + \int_0^b v(t)\,dt = 14.134$ or $14.135$ | $3 : \begin{cases} 1 : \text{considers } v(t) = 0 \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ |

**Question 6**

Two particles move along the $x$-axis. For $0 \le t \le 6$, the position of particle $P$ at time $t$ is given by $p(t) = 2\cos\left(\frac{\pi}{4}t\right)$, while the position of particle $R$ at time $t$ is given by $r(t) = t^3 - 6t^2 + 9t + 3$.

(a) For $0 \le t \le 6$, find all times $t$ during which particle $R$ is moving to the right.

(b) For $0 \le t \le 6$, find all times $t$ during which the two particles travel in opposite directions.

(c) Find the acceleration of particle $P$ at time $t = 3$. Is particle $P$ speeding up, slowing down, or doing neither at time $t = 3$? Explain your reasoning.

(d) Write, but do not evaluate, an expression for the average distance between the two particles on the interval $1 \le t \le 3$.

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| | |
|---|---|
| (a) $r'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$<br>$r'(t) = 0$ when $t = 1$ and $t = 3$<br>$r'(t) > 0$ for $0 < t < 1$ and $3 < t < 6$<br>$r'(t) < 0$ for $1 < t < 3$<br><br>Therefore $R$ is moving to the right for $0 < t < 1$ and $3 < t < 6$. | $2 : \begin{cases} 1 : r'(t) \\ 1 : \text{answer} \end{cases}$ |
| (b) $p'(t) = -2 \cdot \frac{\pi}{4} \sin\left(\frac{\pi}{4}t\right)$<br>$p'(t) = 0$ when $t = 0$ and $t = 4$<br><br>$p'(t) < 0$ for $0 < t < 4$<br>$p'(t) > 0$ for $4 < t < 6$<br><br>Therefore the particles travel in opposite directions for $0 < t < 1$ and $3 < t < 4$. | $3 : \begin{cases} 1 : p'(t) \\ 1 : \text{sign analysis for } p'(t) \\ 1 : \text{answer} \end{cases}$ |
| (c) $p''(t) = -2 \cdot \frac{\pi}{4} \cdot \frac{\pi}{4} \cos\left(\frac{\pi}{4}t\right)$<br>$p''(3) = -2\left(\frac{\pi}{4}\right)^2 \cos\left(\frac{3\pi}{4}\right) = \frac{\pi^2}{8} \cdot \frac{\sqrt{2}}{2} > 0$<br>$p'(3) < 0$<br>Therefore particle $P$ is slowing down at time $t = 3$. | $2 : \begin{cases} 1 : p''(3) \\ 1 : \text{answer with reason} \end{cases}$ |
| (d) $\frac{1}{2}\int_1^3 \lvert p(t) - r(t) \rvert \, dt$ | $2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$ |



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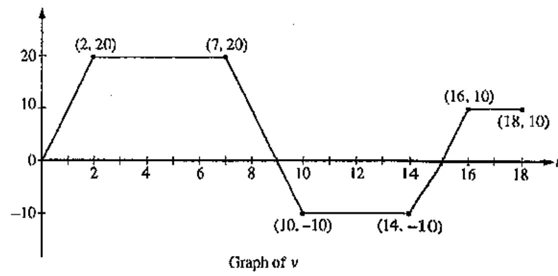
# AP® CALCULUS AB
# 2010 SCORING GUIDELINES (Form B)

## Question 4

A squirrel starts at building $A$ at time $t = 0$ and travels along a straight wire connected to building $B$. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.



Graph of $v$

(a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

(b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building $A$ ? How far from building $A$ is the squirrel at this time?

(c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

(d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building $A$ that are valid for the time interval $7 < t < 10$.

---

(a) The squirrel changes direction whenever its velocity changes sign. This occurs at $t = 9$ and $t = 15$.

$2 : \begin{cases} 1 : t\text{-values} \\ 1 : \text{explanation} \end{cases}$

(b) Velocity is 0 at $t = 0$, $t = 9$, and $t = 15$.

| $t$ | position at time $t$ |
|---|---|
| 0 | 0 |
| 9 | $\frac{9+5}{2} \cdot 20 = 140$ |
| 15 | $140 - \frac{6+4}{2} \cdot 10 = 90$ |
| 18 | $90 + \frac{3+2}{2} \cdot 10 = 115$ |

The squirrel is farthest from building $A$ at time $t = 9$; its greatest distance from the building is 140.

$2 : \begin{cases} 1 : \text{identifies candidates} \\ 1 : \text{answers} \end{cases}$

(c) The total distance traveled is $\int_0^{18} |v(t)|\, dt = 140 + 50 + 25 = 215$.

$1 : \text{answer}$

(d) For $7 < t < 10$, $a(t) = \dfrac{20 - (-10)}{7 - 10} = -10$

$v(t) = 20 - 10(t - 7) = -10t + 90$

$x(7) = \dfrac{7+5}{2} \cdot 20 = 120$

$$x(t) = x(7) + \int_7^t (-10u + 90)\, du$$
$$= 120 + \left(-5u^2 + 90u\right)\Big|_{u=7}^{u=t}$$
$$= -5t^2 + 90t - 265$$

$4 : \begin{cases} 1 : a(t) \\ 1 : v(t) \\ 2 : x(t) \end{cases}$



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## Question 3

A particle moves along the $x$-axis so that its velocity $v$ at time $t$, for $0 \le t \le 5$, is given by $v(t) = \ln\left(t^2 - 3t + 3\right)$. The particle is at position $x = 8$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 4$.

(b) Find all times $t$ in the open interval $0 < t < 5$ at which the particle changes direction. During which time intervals, for $0 \le t \le 5$, does the particle travel to the left?

(c) Find the position of the particle at time $t = 2$.

(d) Find the average speed of the particle over the interval $0 \le t \le 2$.

| | |
|---|---|
| (a) $a(4) = v'(4) = \frac{5}{7}$ | 1 : answer |
| (b) $v(t) = 0$<br>$t^2 - 3t + 3 = 1$<br>$t^2 - 3t + 2 = 0$<br>$(t-2)(t-1) = 0$<br>$t = 1, 2$<br><br>$v(t) > 0$ for $0 < t < 1$<br>$v(t) < 0$ for $1 < t < 2$<br>$v(t) > 0$ for $2 < t < 5$<br><br>The particle changes direction when $t = 1$ and $t = 2$.<br>The particle travels to the left when $1 < t < 2$. | $3 : \begin{cases} 1 : \text{sets } v(t) = 0 \\ 1 : \text{direction change at } t = 1, 2 \\ 1 : \text{interval with reason} \end{cases}$ |
| (c) $s(t) = s(0) + \int_0^t \ln\left(u^2 - 3u + 3\right) du$<br>$s(2) = 8 + \int_0^2 \ln\left(u^2 - 3u + 3\right) du$<br>$= 8.368$ or $8.369$ | $3 : \begin{cases} 1 : \int_0^2 \ln\left(u^2 - 3u + 3\right) du \\ 1 : \text{handles initial condition} \\ 1 : \text{answer} \end{cases}$ |
| (d) $\frac{1}{2}\int_0^2 \lvert v(t)\rvert\, dt = 0.370$ or $0.371$ | $2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ |

### Question 2

A particle moves along the $x$-axis so that its velocity at time $t$ is given by

$$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right).$$

At time $t = 0$, the particle is at position $x = 1$.

(a) Find the acceleration of the particle at time $t = 2$. Is the speed of the particle increasing at $t = 2$? Why or why not?

(b) Find all times $t$ in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.

(c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

(d) During the time interval $0 \le t \le 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

| | |
|---|---|
| (a) $a(2) = v'(2) = 1.587$ or $1.588$<br>$v(2) = -3\sin(2) < 0$<br>Speed is decreasing since $a(2) > 0$ and $v(2) < 0$. | $2: \begin{cases} 1: a(2) \\ 1: \text{speed decreasing} \\ \quad\ \text{with reason} \end{cases}$ |
| (b) $v(t) = 0$ when $\frac{t^2}{2} = \pi$<br>$t = \sqrt{2\pi}$ or $2.506$ or $2.507$<br>Since $v(t) < 0$ for $0 < t < \sqrt{2\pi}$ and $v(t) > 0$ for $\sqrt{2\pi} < t < 3$, the particle changes directions at $t = \sqrt{2\pi}$. | $2: \begin{cases} 1: t = \sqrt{2\pi} \text{ only} \\ 1: \text{justification} \end{cases}$ |
| (c) Distance $= \int_0^3 \lvert v(t) \rvert\, dt = 4.333$ or $4.334$ | $3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$ |
| (d) $\int_0^{\sqrt{2\pi}} v(t)\, dt = -3.265$<br>$x(\sqrt{2\pi}) = x(0) + \int_0^{\sqrt{2\pi}} v(t)\, dt = -2.265$<br>Since the total distance from $t = 0$ to $t = 3$ is 4.334, the particle is still to the left of the origin at $t = 3$. Hence the greatest distance from the origin is 2.265. | $2: \begin{cases} 1: \pm \text{ (distance particle travels} \\ \quad\quad \text{while velocity is negative)} \\ 1: \text{answer} \end{cases}$ |





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## Question 3

An object moves along the $x$-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

(a) What is the acceleration of the object at time $t = 4$?

(b) Consider the following two statements.

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.

Statement II: For $3 < t < 4.5$, the speed of the object is increasing.

Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

(c) What is the total distance traveled by the object over the time interval $0 \leq t \leq 4$?

(d) What is the position of the object at time $t = 4$?

| Solution | Scoring |
|---|---|
| (a) $a(4) = v'(4) = \frac{\pi}{3}\cos\left(\frac{4\pi}{3}\right) = -\frac{\pi}{6}$ or $-0.523$ or $-0.524$ | 1 : answer |
| (b) On $3 < t < 4.5$:<br>$a(t) = v'(t) = \frac{\pi}{3}\cos\left(\frac{\pi}{3}t\right) < 0$<br>Statement I is correct since $a(t) < 0$.<br>Statement II is correct since $v(t) < 0$ and $a(t) < 0$. | 3 { 1 : I correct, with reason<br>1 : II correct<br>1 : reason for II |
| (c) Distance $= \int_0^4 \lvert v(t)\rvert\,dt = 2.387$<br>OR<br>$x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi} + 2$<br>$x(0) = 2$<br>$x(4) = 2 + \frac{9}{2\pi} = 3.43239$<br>$v(t) = 0$ when $t = 3$<br>$x(3) = \frac{6}{\pi} + 2 = 3.90986$<br>$\lvert x(3) - x(0)\rvert + \lvert x(4) - x(3)\rvert = \frac{15}{2\pi} = 2.387$ | 3 { 1 : { limits of 0 and 4 on an integral of $v(t)$ or $\lvert v(t)\rvert$ or uses $x(0)$ and $x(4)$ to compute distance<br>1 : handles change of direction at student's turning point<br>1 : answer<br>0/1 if incorrect turning point or no turning point |
| (d) $x(4) = x(0) + \int_0^4 v(t)\,dt = 3.432$<br>OR<br>$x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi} + 2$<br>$x(4) = 2 + \frac{9}{2\pi} = 3.432$ | 2 { 1 : integral<br>1 : answer<br>OR<br>2 { 1 : $x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + C$<br>1 : answer<br>0/1 if no constant of integration |





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### Question 6

| $t$ (sec) | 0 | 15 | 25 | 30 | 35 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $v(t)$ (ft/sec) | −20 | −30 | −20 | −14 | −10 | 0 | 10 |
| $a(t)$ $\left(\text{ft}/\text{sec}^2\right)$ | 1 | 5 | 2 | 1 | 2 | 4 | 2 |

A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity $v$, measured in feet per second, and acceleration $a$, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

(a) Using appropriate units, explain the meaning of $\int_{30}^{60} |v(t)|\, dt$ in terms of the car's motion. Approximate $\int_{30}^{60} |v(t)|\, dt$ using a trapezoidal approximation with the three subintervals determined by the table.

(b) Using appropriate units, explain the meaning of $\int_{0}^{30} a(t)\, dt$ in terms of the car's motion. Find the exact value of $\int_{0}^{30} a(t)\, dt$.

(c) For $0 < t < 60$, must there be a time $t$ when $v(t) = -5$ ? Justify your answer.

(d) For $0 < t < 60$, must there be a time $t$ when $a(t) = 0$ ? Justify your answer.

---

(a) $\int_{30}^{60} |v(t)|\, dt$ is the distance in feet that the car travels from $t = 30$ sec to $t = 60$ sec.

Trapezoidal approximation for $\int_{30}^{60} |v(t)|\, dt$ :

$$A = \frac{1}{2}(14+10)5 + \frac{1}{2}(10)(15) + \frac{1}{2}(10)(10) = 185 \text{ ft}$$

$2 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{value} \end{cases}$

(b) $\int_{0}^{30} a(t)\, dt$ is the car's change in velocity in ft/sec from $t = 0$ sec to $t = 30$ sec.

$$\int_{0}^{30} a(t)\, dt = \int_{0}^{30} v'(t)\, dt = v(30) - v(0)$$
$$= -14 - (-20) = 6 \text{ ft/sec}$$

$2 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{value} \end{cases}$

(c) Yes. Since $v(35) = -10 < -5 < 0 = v(50)$, the IVT guarantees a $t$ in $(35, 50)$ so that $v(t) = -5$.

$2 : \begin{cases} 1 : v(35) < -5 < v(50) \\ 1 : \text{Yes; refers to IVT or hypotheses} \end{cases}$

(d) Yes. Since $v(0) = v(25)$, the MVT guarantees a $t$ in $(0, 25)$ so that $a(t) = v'(t) = 0$.

$2 : \begin{cases} 1 : v(0) = v(25) \\ 1 : \text{Yes; refers to MVT or hypotheses} \end{cases}$

Units of ft in (a) and ft/sec in (b)

1 : units in (a) and (b)





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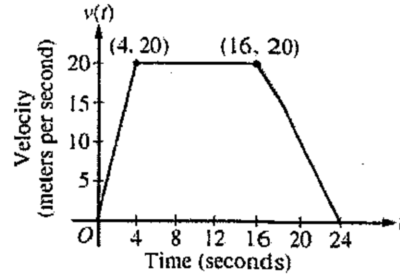
### Question 5

A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.




(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time $t$, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of $v$ over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of $c$, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

---

(a) $\int_0^{24} v(t)\,dt = \frac{1}{2}(4)(20) + (12)(20) + \frac{1}{2}(8)(20) = 360$

The car travels 360 meters in these 24 seconds.

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning with units} \end{cases}$

(b) $v'(4)$ does not exist because

$$\lim_{t\to 4^-}\left(\frac{v(t)-v(4)}{t-4}\right) = 5 \ne 0 = \lim_{t\to 4^+}\left(\frac{v(t)-v(4)}{t-4}\right).$$

$$v'(20) = \frac{20-0}{16-24} = -\frac{5}{2}\ \text{m/sec}^2$$

$3 : \begin{cases} 1 : v'(4) \text{ does not exist, with explanation} \\ 1 : v'(20) \\ 1 : \text{units} \end{cases}$

(c)

$$a(t) = \begin{cases} 5 & \text{if } 0 < t < 4 \\ 0 & \text{if } 4 < t < 16 \\ -\frac{5}{2} & \text{if } 16 < t < 24 \end{cases}$$

$a(t)$ does not exist at $t = 4$ and $t = 16$.

$2 : \begin{cases} 1 : \text{finds the values } 5, 0, -\frac{5}{2} \\ 1 : \text{identifies constants with correct intervals} \end{cases}$

(d) The average rate of change of $v$ on $[8, 20]$ is

$$\frac{v(20)-v(8)}{20-8} = -\frac{5}{6}\ \text{m/sec}^2.$$

No, the Mean Value Theorem does not apply to $v$ on $[8, 20]$ because $v$ is not differentiable at $t = 16$.

$2 : \begin{cases} 1 : \text{average rate of change of } v \text{ on } [8, 20] \\ 1 : \text{answer with explanation} \end{cases}$





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## Question 5

| $t$ (seconds) | 0 | 10 | 40 | 60 |
|---|---|---|---|---|
| $B(t)$ (meters) | 100 | 136 | 9 | 49 |
| $v(t)$ (meters per second) | 2.0 | 2.3 | 2.5 | 4.6 |

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function $B$ models Ben's position on the track, measured in meters from the western end of the track, at time $t$, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times $t$.

(a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\,dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\,dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

(c) For $40 \le t \le 60$, must there be a time $t$ when Ben's velocity is 2 meters per second? Justify your answer.

(d) A light is directly above the western end of the track. Ben rides so that at time $t$, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$ ?

---

| | |
|---|---|
| (a) $a(5) \approx \dfrac{v(10) - v(0)}{10 - 0} = \dfrac{0.3}{10} = 0.03$ meters/sec$^2$ | 1 : answer |
| (b) $\int_0^{60} |v(t)|\,dt$ is the total distance, in meters, that Ben rides over the 60-second interval $t = 0$ to $t = 60$. <br><br> $\int_0^{60} |v(t)|\,dt \approx 2.0 \cdot 10 + 2.3(40 - 10) + 2.5(60 - 40) = 139$ meters | 2 : $\begin{cases} 1 : \text{meaning of integral} \\ 1 : \text{approximation} \end{cases}$ |
| (c) Because $\dfrac{B(60) - B(40)}{60 - 40} = \dfrac{49 - 9}{20} = 2$, the Mean Value Theorem implies there is a time $t$, $40 < t < 60$, such that $v(t) = 2$. | 2 : $\begin{cases} 1 : \text{difference quotient} \\ 1 : \text{conclusion with justification} \end{cases}$ |
| (d) $2L(t)L'(t) = 2B(t)B'(t)$ <br> $L'(40) = \dfrac{B(40)v(40)}{L(40)} = \dfrac{9 \cdot 2.5}{\sqrt{144 + 81}} = \dfrac{3}{2}$ meters/sec | 3 : $\begin{cases} 1 : \text{derivatives} \\ 1 : \text{uses } B'(t) = v(t) \\ 1 : \text{answer} \end{cases}$ |
| | 1 : units in (a) or (b) |



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# AP® CALCULUS AB
# 2009 SCORING GUIDELINES (Form B)

## Question 6

| $t$ (seconds) | 0 | 8 | 20 | 25 | 32 | 40 |
|---|---|---|---|---|---|---|
| $v(t)$ (meters per second) | 3 | 5 | $-10$ | $-8$ | $-4$ | 7 |

The velocity of a particle moving along the $x$-axis is modeled by a differentiable function $v$, where the position $x$ is measured in meters, and time $t$ is measured in seconds. Selected values of $v(t)$ are given in the table above. The particle is at position $x = 7$ meters when $t = 0$ seconds.

(a) Estimate the acceleration of the particle at $t = 36$ seconds. Show the computations that lead to your answer. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{20}^{40} v(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate $\int_{20}^{40} v(t)\,dt$.

(c) For $0 \le t \le 40$, must the particle change direction in any of the subintervals indicated by the data in the table? If so, identify the subintervals and explain your reasoning. If not, explain why not.

(d) Suppose that the acceleration of the particle is positive for $0 < t < 8$ seconds. Explain why the position of the particle at $t = 8$ seconds must be greater than $x = 30$ meters.

---

1 : units in (a) and (b)

(a) $a(36) = v'(36) \approx \dfrac{v(40) - v(32)}{40 - 32} = \dfrac{11}{8}$ meters/sec$^2$

1 : answer

(b) $\int_{20}^{40} v(t)\,dt$ is the particle's change in position in meters from time $t = 20$ seconds to time $t = 40$ seconds.

$$\int_{20}^{40} v(t)\,dt \approx \frac{v(20) + v(25)}{2} \cdot 5 + \frac{v(25) + v(32)}{2} \cdot 7 + \frac{v(32) + v(40)}{2} \cdot 8$$
$$= -75 \text{ meters}$$

$3 : \begin{cases} 1 : \text{meaning of } \int_{20}^{40} v(t)\,dt \\ 2 : \text{trapezoidal approximation} \end{cases}$

(c) $v(8) > 0$ and $v(20) < 0$
$v(32) < 0$ and $v(40) > 0$
Therefore, the particle changes direction in the intervals $8 < t < 20$ and $32 < t < 40$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

(d) Since $v'(t) = a(t) > 0$ for $0 < t < 8$, $v(t) \ge 3$ on this interval.
Therefore, $x(8) = x(0) + \int_0^8 v(t)\,dt \ge 7 + 8 \cdot 3 > 30$.

$2 : \begin{cases} 1 : v'(t) = a(t) \\ 1 : \text{explanation of } x(8) > 30 \end{cases}$

### Question 4

| $t$ (seconds) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|---|
| $v(t)$ (feet per second) | 5 | 14 | 22 | 29 | 35 | 40 | 44 | 47 | 49 |

Rocket $A$ has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of $t$ over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket $A$ over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

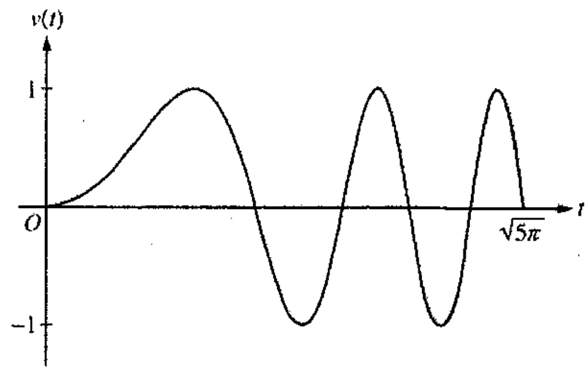
(c) Rocket $B$ is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

| | |
|---|---|
| (a) Average acceleration of rocket $A$ is $\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20}\ \text{ft}/\text{sec}^2$ | 1 : answer |
| (b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket $A$ from $t = 10$ seconds to $t = 70$ seconds. A midpoint Riemann sum is $20[v(20) + v(40) + v(60)] = 20[22 + 35 + 44] = 2020$ ft | $3 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{uses } v(20),\ v(40),\ v(60) \\ 1 : \text{value} \end{cases}$ |
| (c) Let $v_B(t)$ be the velocity of rocket $B$ at time $t$. $v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$; $2 = v_B(0) = 6 + C$; $v_B(t) = 6\sqrt{t+1} - 4$; $v_B(80) = 50 > 49 = v(80)$. Rocket $B$ is traveling faster at time $t = 80$ seconds. | $4 : \begin{cases} 1 : 6\sqrt{t+1} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{finds } v_B(80) \text{, compares to } v(80) \text{,} \\ \quad \text{and draws a conclusion} \end{cases}$ |
| Units of $\text{ft}/\text{sec}^2$ in (a) and ft in (b) | 1 : units in (a) and (b) |

# AP® CALCULUS AB
# 2007 SCORING GUIDELINES (Form B)

## Question 2

A particle moves along the $x$-axis so that its velocity $v$ at time $t \ge 0$ is given by $v(t) = \sin(t^2)$. The graph of $v$ is shown above for $0 \le t \le \sqrt{5\pi}$. The position of the particle at time $t$ is $x(t)$ and its position at time $t = 0$ is $x(0) = 5$.




(a) Find the acceleration of the particle at time $t = 3$.

(b) Find the total distance traveled by the particle from time $t = 0$ to $t = 3$.

(c) Find the position of the particle at time $t = 3$.

(d) For $0 \le t \le \sqrt{5\pi}$, find the time $t$ at which the particle is farthest to the right. Explain your answer.

---

(a) $a(3) = v'(3) = 6\cos 9 = -5.466$ or $-5.467$

$1: a(3)$

(b) Distance $= \int_0^3 |v(t)|\,dt = 1.702$

OR

For $0 < t < 3$, $v(t) = 0$ when $t = \sqrt{\pi} = 1.77245$ and $t = \sqrt{2\pi} = 2.50663$

$x(0) = 5$

$x(\sqrt{\pi}) = 5 + \int_0^{\sqrt{\pi}} v(t)\,dt = 5.89483$

$x(\sqrt{2\pi}) = 5 + \int_0^{\sqrt{2\pi}} v(t)\,dt = 5.43041$

$x(3) = 5 + \int_0^3 v(t)\,dt = 5.77356$

$|x(\sqrt{\pi}) - x(0)| + |x(\sqrt{2\pi}) - x(\sqrt{\pi})| + |x(3) - x(\sqrt{2\pi})| = 1.702$

$2: \begin{cases} 1: \text{setup} \\ 1: \text{answer} \end{cases}$

---

(c) $x(3) = 5 + \int_0^3 v(t)\,dt = 5.773$ or $5.774$

$3: \begin{cases} 2 \begin{cases} 1: \text{integrand} \\ 1: \text{uses } x(0) = 5 \end{cases} \\ 1: \text{answer} \end{cases}$

(d) The particle's rightmost position occurs at time $t = \sqrt{\pi} = 1.772$.

The particle changes from moving right to moving left at those times $t$ for which $v(t) = 0$ with $v(t)$ changing from positive to negative, namely at $t = \sqrt{\pi}, \sqrt{3\pi}, \sqrt{5\pi}$ ($t = 1.772, 3.070, 3.963$).

Using $x(T) = 5 + \int_0^T v(t)\,dt$, the particle's positions at the times it changes from rightward to leftward movement are:

| $T$: | 0 | $\sqrt{\pi}$ | $\sqrt{3\pi}$ | $\sqrt{5\pi}$ |
|---|---|---|---|---|
| $x(T)$: | 5 | 5.895 | 5.788 | 5.752 |

The particle is farthest to the right when $T = \sqrt{\pi}$.

$3: \begin{cases} 1: \text{sets } v(t) = 0 \\ 1: \text{answer} \\ 1: \text{reason} \end{cases}$

# AP® CALCULUS AB
# 2002 SCORING GUIDELINES (Form B)

## Question 3

A particle moves along the $x$-axis so that its velocity $v$ at any time $t$, for $0 \le t \le 16$, is given by $v(t) = e^{2\sin t} - 1$. At time $t = 0$, the particle is at the origin.

(a) On the axes provided, sketch the graph of $v(t)$ for $0 \le t \le 16$.

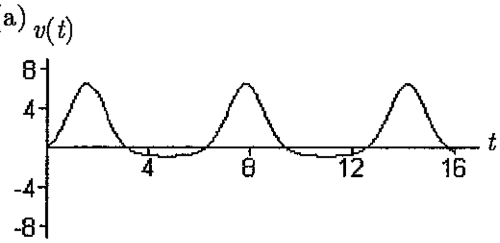
(b) During what intervals of time is the particle moving to the left? Give a reason for your answer.

(c) Find the total distance traveled by the particle from $t = 0$ to $t = 4$.

(d) Is there any time $t$, $0 < t \le 16$, at which the particle returns to the origin? Justify your answer.

---

(a)




1 : graph
- three "humps"
- periodic behavior
- starts at origin
- reasonable relative max and min values

(b) Particle is moving to the left when $v(t) < 0$, i.e. $e^{2\sin t} < 1$.

$(\pi, 2\pi)$, $(3\pi, 4\pi)$ and $(5\pi, 16]$

$$3 \begin{cases} 2 : \text{intervals} \\ \quad < -1 > \text{ each missing or incorrect interval} \\ 1 : \text{reason} \end{cases}$$

(c) $\int_0^4 |v(t)|\,dt = 10.542$

or

$v(t) = e^{2\sin t} - 1 = 0$

$t = 0$ or $t = \pi$

$x(\pi) = \int_0^{\pi} v(t)\,dt = 10.10656$

$x(4) = \int_0^{4} v(t)\,dt = 9.67066$

$|x(\pi) - x(0)| + |x(4) - x(\pi)|$

$= 10.542$

$$3 \begin{cases} 1 : \text{limits of 0 and 4 on an integral of} \\ \quad v(t) \text{ or } |v(t)| \\ \quad \text{or} \\ \quad \text{uses } x(0) \text{ and } x(4) \text{ to compute distance} \\ 1 : \text{handles change of direction at student's} \\ \quad \text{turning point} \\ 1 : \text{answer} \\ \quad \text{note: 0/1 if incorrect turning point} \end{cases}$$

(d) There is no such time because $\int_0^T v(t)\,dt > 0$ for all $T > 0$.

$$2 \begin{cases} 1 : \text{no such time} \\ 1 : \text{reason} \end{cases}$$





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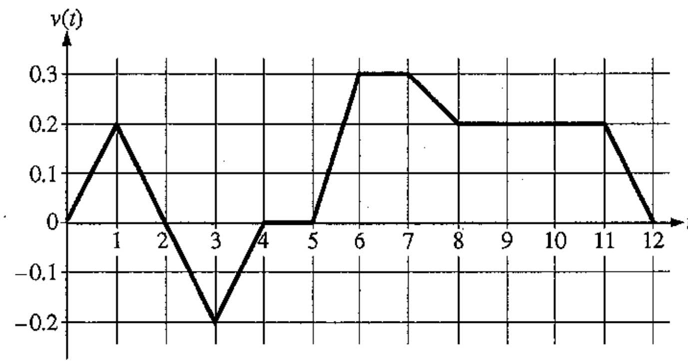
## Question 1




Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function $w$ given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

| | |
|---|---|
| (a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = -0.1$ miles/minute$^2$ | $2: \begin{cases} 1: \text{answer} \\ 1: \text{units} \end{cases}$ |
| (b) $\int_0^{12} \lvert v(t)\rvert\, dt$ is the total distance, in miles, that Caren rode during the 12 minutes from $t = 0$ to $t = 12$.<br>$\int_0^{12} \lvert v(t)\rvert\, dt = \int_0^2 v(t)\, dt - \int_2^4 v(t)\, dt + \int_4^{12} v(t)\, dt$<br>$= 0.2 + 0.2 + 1.4 = 1.8$ miles | $2: \begin{cases} 1: \text{meaning of integral} \\ 1: \text{value of integral} \end{cases}$ |
| (c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative. | $2: \begin{cases} 1: \text{answer} \\ 1: \text{reason} \end{cases}$ |
| (d) $\int_0^{12} w(t)\, dt = 1.6$; Larry lives 1.6 miles from school.<br>$\int_0^{12} v(t)\, dt = 1.4$; Caren lives 1.4 miles from school.<br>Therefore, Caren lives closer to school. | $3: \begin{cases} 2: \text{Larry's distance from school} \\ \quad 1: \text{integral} \\ \quad 1: \text{value} \\ 1: \text{Caren's distance from school and conclusion} \end{cases}$ |





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### Question 3

A particle moves along the $y$-axis so that its velocity $v$ at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}\left(e^{t}\right)$. At time $t = 0$, the particle is at $y = -1$. (Note: $\tan^{-1} x = \arctan x$)

(a) Find the acceleration of the particle at time $t = 2$.

(b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

(c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

(d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

| | |
|---|---|
| (a) $a(2) = v'(2) = -0.132$ or $-0.133$ | 1 : answer |
| (b) $v(2) = -0.436$<br>Speed is increasing since $a(2) < 0$ and $v(2) < 0$. | 1 : answer with reason |
| (c) $v(t) = 0$ when $\tan^{-1}\left(e^{t}\right) = 1$<br>$t = \ln(\tan(1)) = 0.443$ is the only critical value for $y$.<br><br>$v(t) > 0$ for $0 < t < \ln(\tan(1))$<br>$v(t) < 0$ for $t > \ln(\tan(1))$<br><br>$y(t)$ has an absolute maximum at $t = 0.443$. | 3 : $\begin{cases} 1 : \text{sets } v(t) = 0 \\ 1 : \text{identifies } t = 0.443 \text{ as a candidate} \\ 1 : \text{justifies absolute maximum} \end{cases}$ |
| (d) $y(2) = -1 + \int_0^2 v(t)\,dt = -1.360$ or $-1.361$<br>The particle is moving away from the origin since $v(2) < 0$ and $y(2) < 0$. | 4 : $\begin{cases} 1 : \int_0^2 v(t)\,dt \\ 1 : \text{handles initial condition} \\ 1 : \text{value of } y(2) \\ 1 : \text{answer with reason} \end{cases}$ |





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## Question 4

A particle moves along the $x$-axis with position at time $t$ given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$.

(a) Find the time $t$ at which the particle is farthest to the left. Justify your answer.

(b) Find the value of the constant $A$ for which $x(t)$ satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$ for $0 < t < 2\pi$.

---

(a) $x'(t) = -e^{-t}\sin t + e^{-t}\cos t = e^{-t}(\cos t - \sin t)$

$x'(t) = 0$ when $\cos t = \sin t$. Therefore, $x'(t) = 0$ on $0 \le t \le 2\pi$ for $t = \frac{\pi}{4}$ and $t = \frac{5\pi}{4}$.

The candidates for the absolute minimum are at $t = 0, \frac{\pi}{4}, \frac{5\pi}{4}$, and $2\pi$.

| $t$ | $x(t)$ |
|---|---|
| $0$ | $e^0\sin(0) = 0$ |
| $\frac{\pi}{4}$ | $e^{-\frac{\pi}{4}}\sin\left(\frac{\pi}{4}\right) > 0$ |
| $\frac{5\pi}{4}$ | $e^{-\frac{5\pi}{4}}\sin\left(\frac{5\pi}{4}\right) < 0$ |
| $2\pi$ | $e^{-2\pi}\sin(2\pi) = 0$ |

The particle is farthest to the left when $t = \frac{5\pi}{4}$.

$$5 : \begin{cases} 2 : x'(t) \\ 1 : \text{sets } x'(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

---

(b) $x''(t) = -e^{-t}(\cos t - \sin t) + e^{-t}(-\sin t - \cos t)$
$= -2e^{-t}\cos t$

$Ax''(t) + x'(t) + x(t)$
$= A\left(-2e^{-t}\cos t\right) + e^{-t}(\cos t - \sin t) + e^{-t}\sin t$
$= (-2A + 1)e^{-t}\cos t$
$= 0$

Therefore, $A = \frac{1}{2}$.

$$4 : \begin{cases} 2 : x''(t) \\ 1 : \text{substitutes } x''(t),\ x'(t), \text{ and } x(t) \\ \quad \text{into } Ax''(t) + x'(t) + x(t) \\ 1 : \text{answer} \end{cases}$$

# Area between Curves

93 BC 1. The area of the region enclosed by the graphs of $y = x^2$ and $y = x$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{5}{6}$ (E) 1

85 BC 1. The area of the region between the graph of $y = 4x^3 + 2$ and the $x$-axis from $x = 1$ to $x = 2$ is

(A) 36 (B) 23 (C) 20 (D) 17 (E) 9

88 21. The area of the region enclosed by the graphs of $y = x$ and $y = x^2 - 3x + 3$ is

(A) $\frac{2}{3}$ (B) 1 (C) $\frac{4}{3}$ (D) 2 (E) $\frac{14}{3}$

98 25. What is the area of the region between the graphs of $y = x^2$ and $y = -x$ from $x = 0$ to $x = 2$?

(A) $\frac{2}{3}$ (B) $\frac{8}{3}$ (C) 4 (D) $\frac{14}{3}$ (E) $\frac{16}{3}$

69 AB 23. The area of the region bounded by the curve $y = e^{2x}$, the $x$-axis, the $y$-axis, and the line $x = 2$ is equal to

(A) $\frac{e^4}{2} - e$ (B) $\frac{e^4}{2} - 1$ (C) $\frac{e^4}{2} - \frac{1}{2}$

(D) $2e^4 - e$ (E) $2e^4 - 2$

97 16. The area of the region enclosed by the graph of $y = x^2 + 1$ and the line $y = 5$ is

(A) $\frac{14}{3}$ (B) $\frac{16}{3}$ (C) $\frac{28}{3}$ (D) $\frac{32}{3}$ (E) $8\pi$

85 34. The area of the region in the first quadrant that is enclosed by the graphs of $y = x^3 + 8$ and $y = x + 8$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1 (E) $\frac{65}{4}$


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73 17. What is the area of the region completely bounded by the curve $y=-x^2+x+6$ and the line $y=4$?

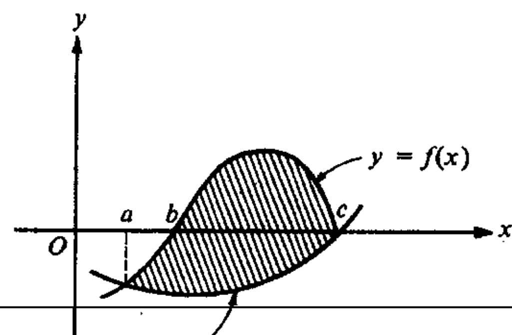
(A) $\frac{3}{2}$ (B) $\frac{7}{3}$ (C) $\frac{9}{2}$ (D) $\frac{31}{6}$ (E) $\frac{33}{2}$

93 6. The area of the region enclosed by the curve $y=\frac{1}{x-1}$, the $x$-axis, and the lines $x=3$ and $x=4$ is

(A) $\frac{5}{36}$ (B) $\ln\frac{2}{3}$ (C) $\ln\frac{4}{3}$ (D) $\ln\frac{3}{2}$ (E) $\ln 6$

88 BC 1. The area of the region in the first quadrant enclosed by the graph of $y=x(1-x)$ and the $x$-axis is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{5}{6}$ (E) 1




88 34. The area of the shaded region in the figure above is represented by which of the following integrals?

(A) $\int_a^c (|f(x)|-|g(x)|)\,dx$

(B) $\int_b^c f(x)\,dx-\int_a^c g(x)\,dx$

(C) $\int_a^c (g(x)-f(x))\,dx$

(D) $\int_a^c (f(x)-g(x))\,dx$

(E) $\int_a^b (g(x)-f(x))\,dx+\int_b^c (f(x)-g(x))\,dx$

Calculator Active

97 83. What is the area of the region in the first quadrant enclosed by the graphs of $y = \cos x$, $y = x$, and the $y$-axis?

(A) 0.127 (B) 0.385 (C) 0.400 (D) 0.600 (E) 0.947

98 92. If $0 \le k < \frac{\pi}{2}$ and the area under the curve $y = \cos x$ from $x = k$ to $x = \frac{\pi}{2}$ is 0.1, then $k =$

(A) 1.471 (B) 1.414 (C) 1.277 (D) 1.120 (E) 0.436

98 BC 80. Let $R$ be the region enclosed by the graph of $y = 1 + \ln\left(\cos^4 x\right)$, the $x$-axis, and the lines $x = -\frac{2}{3}$ and $x = \frac{2}{3}$. The closest integer approximation of the area of $R$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4



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69 AB 25. A region in the plane is bounded by the graph of $y = \frac{1}{x}$, the $x$-axis, the line $x = m$, and the line $x = 2m$, $m > 0$. The area of this region

(A) is independent of $m$.

(B) increases as $m$ increases.

(C) decreases as $m$ increases.

(D) decreases as $m$ increases when $m < \frac{1}{2}$; increases as $m$ increases when $m > \frac{1}{2}$.

(E) increases as $m$ increases when $m < \frac{1}{2}$; decreases as $m$ increases when $m > \frac{1}{2}$.

---

64 AB 13. The region bounded by the $x$-axis and the part of the graph of $y = \cos x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is separated into two regions by the line $x = k$. If the area of the region for $-\frac{\pi}{2} \le x \le k$ is three times the area of the region for $k \le x \le \frac{\pi}{2}$, then $k =$

(A) $\arcsin\left(\frac{1}{4}\right)$

(B) $\arcsin\left(\frac{1}{3}\right)$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{4}$

(E) $\frac{\pi}{3}$



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# Volumes and Rotationals

## Disk Method

93 30. The region enclosed by the $x$-axis, the line $x = 3$, and the curve $y = \sqrt{x}$ is rotated about the $x$-axis. What is the volume of the solid generated?

(A) $3\pi$ (B) $2\sqrt{3}\,\pi$ (C) $\frac{9}{2}\pi$ (D) $9\pi$ (E) $\frac{36\sqrt{3}}{5}\pi$

97 23. If the region enclosed by the $y$-axis, the line $y = 2$, and the curve $y = \sqrt{x}$ is revolved about the $y$-axis, the volume of the solid generated is

(A) $\frac{32\pi}{5}$ (B) $\frac{16\pi}{3}$ (C) $\frac{16\pi}{5}$ (D) $\frac{8\pi}{3}$ (E) $\pi$

93 BC 30. What is the volume of the solid generated by rotating about the $x$-axis the region enclosed by the curve $y = \sec x$ and the lines $x = 0$, $y = 0$, and $x = \frac{\pi}{3}$?

(A) $\frac{\pi}{\sqrt{3}}$

(B) $\pi$

(C) $\pi\sqrt{3}$

(D) $\frac{8\pi}{3}$

(E) $\pi \ln\left(\frac{1}{2} + \sqrt{3}\right)$

88 BC 36. Let $R$ be the region between the graphs of $y = 1$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$. The volume of the solid obtained by revolving $R$ about the x-axis is given by

(A) $2\pi \int_0^{\frac{\pi}{2}} x \sin x \, dx$ (B) $2\pi \int_0^{\frac{\pi}{2}} x \cos x \, dx$ (C) $\pi \int_0^{\frac{\pi}{2}} (1 - \sin x)^2 \, dx$

(D) $\pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ (E) $\pi \int_0^{\frac{\pi}{2}} \left(1 - \sin^2 x\right) dx$

88 BC 29. The region $R$ in the first quadrant is enclosed by the lines $x = 0$ and $y = 5$ and the graph of $y = x^2 + 1$. The volume of the solid generated when $R$ is revolved about the $y$-axis is

(A) $6\pi$  (B) $8\pi$  (C) $\frac{34\pi}{3}$  (D) $16\pi$  (E) $\frac{544\pi}{15}$

## Washer Method

85 45. The region enclosed by the graph of $y = x^2$, the line $x = 2$, and the $x$-axis is revolved about the $y$-axis. The volume of the solid generated is

(A) $8\pi$  (B) $\frac{32}{5}\pi$  (C) $\frac{16}{3}\pi$  (D) $4\pi$  (E) $\frac{8}{3}\pi$

73 35. The region in the first quadrant bounded by the graph of $y = \sec x$, $x = \frac{\pi}{4}$, and the axes is rotated about the $x$-axis. What is the volume of the solid generated?

(A) $\frac{\pi^2}{4}$  (B) $\pi - 1$  (C) $\pi$  (D) $2\pi$  (E) $\frac{8\pi}{3}$

# Rotationals - Disk/Washer

85 BC 35. The region in the first quadrant between the $x$-axis and the graph of $y = 6x - x^2$ is rotated around the $y$-axis. The volume of the resulting solid of revolution is given by

(A) $\int_0^6 \pi\left(6x - x^2\right)^2 dx$

(B) $\int_0^6 2\pi x\left(6x - x^2\right) dx$

(C) $\int_0^6 \pi x\left(6x - x^2\right)^2 dx$

(D) $\int_0^6 \pi\left(3 + \sqrt{9-y}\right)^2 dy$

(E) $\int_0^9 \pi\left(3 + \sqrt{9-y}\right)^2 dy$

97 BC 77. When the region enclosed by the graphs of $y = x$ and $y = 4x - x^2$ is revolved about the $y$-axis, the volume of the solid generated is given by

(A) $\pi\int_0^3 \left(x^3 - 3x^2\right) dx$

(B) $\pi\int_0^3 \left(x^2 - \left(4x - x^2\right)^2\right) dx$

(C) $\pi\int_0^3 \left(3x - x^2\right)^2 dx$

(D) $2\pi\int_0^3 \left(x^3 - 3x^2\right) dx$

(E) $2\pi\int_0^3 \left(3x^2 - x^3\right) dx$

93 20. Let $R$ be the region in the first quadrant enclosed by the graph of $y = (x+1)^{\frac{1}{3}}$, the line $x = 7$, the $x$-axis, and the $y$-axis. The volume of the solid generated when $R$ is revolved about the $y$-axis is given by

(A) $\pi\int_0^7 (x+1)^{\frac{2}{3}} dx$

(B) $2\pi\int_0^7 x(x+1)^{\frac{1}{3}} dx$

(C) $\pi\int_0^2 (x+1)^{\frac{2}{3}} dx$

(D) $2\pi\int_0^2 x(x+1)^{\frac{1}{3}} dx$

(E) $\pi\int_0^7 (y^3 - 1)^2 dy$

88 30. A region in the first quadrant is enclosed by the graphs of $y = e^{2x}$, $x = 1$, and the coordinate axes. If the region is rotated about the $y$-axis, the volume of the solid that is generated is represented by which of the following integrals?

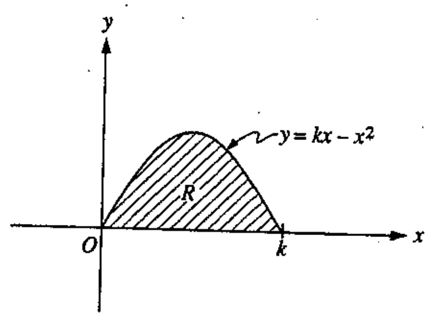
(A) $2\pi \int_0^1 xe^{2x}\,dx$

(B) $2\pi \int_0^1 e^{2x}\,dx$

(C) $\pi \int_0^1 e^{4x}\,dx$

(D) $\pi \int_0^e y \ln y\,dy$

(E) $\frac{\pi}{4} \int_0^e \ln^2 y\,dy$




93 BC 19. ~~The shaded region~~ $R$, shown in the figure above, is rotated about the $y$-axis to form a solid whose volume is 10 cubic units. Of the following, which best approximates $k$?

(A) 1.51 (B) 2.09 (C) 2.49 (D) 4.18 (E) 4.77

Rotations - Cross Sections

85 BC

39. The base of a solid is the region enclosed by the graph of $y = e^{-x}$, the coordinate axes, and the line $x = 3$. If all plane cross sections perpendicular to the $x$-axis are squares, then its volume is

(A) $\dfrac{(1-e^{-6})}{2}$ (B) $\dfrac{1}{2}e^{-6}$ (C) $e^{-6}$ (D) $e^{-3}$ (E) $1-e^{-3}$

97 BC

87. The base of a solid is the region in the first quadrant enclosed by the graph of $y = 2 - x^2$ and the coordinate axes. If every cross section of the solid perpendicular to the $y$-axis is a square, the volume of the solid is given by

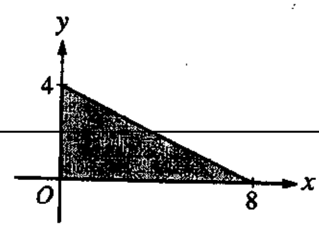
(A) $\pi\int_0^2 (2-y)^2\,dy$

(B) $\int_0^2 (2-y)\,dy$

(C) $\pi\int_0^{\sqrt{2}} \left(2-x^2\right)^2 dx$

(D) $\int_0^{\sqrt{2}} \left(2-x^2\right)^2 dx$

(E) $\int_0^{\sqrt{2}} \left(2-x^2\right) dx$




98 BC

86. The base of a solid is a region in the first quadrant bounded by the $x$-axis, the $y$-axis, and the line $x + 2y = 8$, as shown in the figure above. If cross sections of the solid perpendicular to the $x$-axis are semicircles, what is the volume of the solid?

(A) 12.566 (B) 14.661 (C) 16.755 (D) 67.021 (E) 134.041

88 BC

25. The base of a solid is the region in the first quadrant enclosed by the parabola $y = 4x^2$, the line $x = 1$, and the $x$-axis. Each plane section of the solid perpendicular to the $x$-axis is a square. The volume of the solid is

(A) $\dfrac{4\pi}{3}$ (B) $\dfrac{16\pi}{5}$ (C) $\dfrac{4}{3}$ (D) $\dfrac{16}{5}$ (E) $\dfrac{64}{5}$

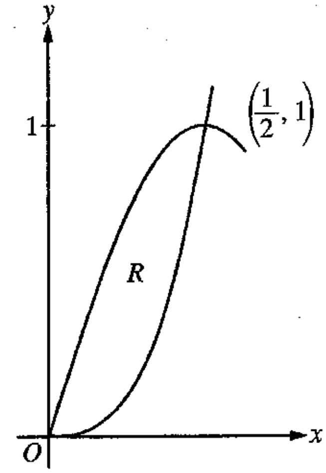
250 250

97 84. The base of a solid $S$ is the region enclosed by the graph of $y = \sqrt{\ln x}$, the line $x = e$, and the $x$-axis. If the cross sections of $S$ perpendicular to the $x$-axis are squares, then the volume of $S$ is

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) 1 (D) 2 (E) $\frac{1}{3}(e^3 - 1)$

# Area between Curves / Volume of Solids FRQ

**NO CALCULATOR**



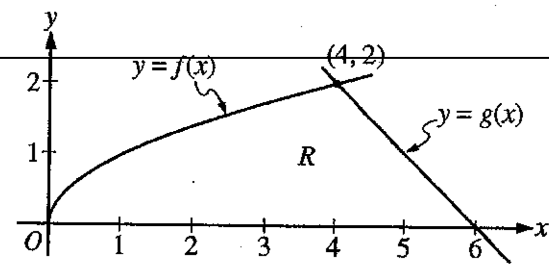

2011

3. Let $R$ be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of $f$ at $x = \frac{1}{2}$.

(b) Find the area of $R$.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when $R$ is rotated about the horizontal line $y = 1$.



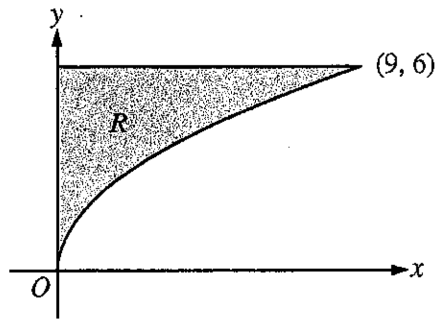

11-B

3. The functions $f$ and $g$ are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let $R$ be the region bounded by the $x$-axis and the graphs of $f$ and $g$, as shown in the figure above.

(a) Find the area of $R$.

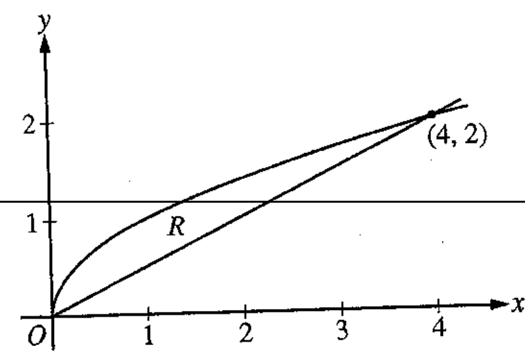
(b) The region $R$ is the base of a solid. For each $y$, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the $y$-axis is a rectangle whose base lies in $R$ and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(c) There is a point $P$ on the graph of $f$ at which the line tangent to the graph of $f$ is perpendicular to the graph of $g$. Find the coordinates of point $P$.



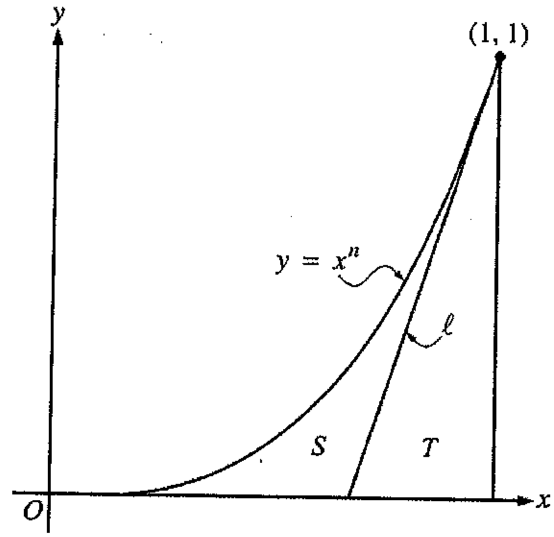

10-A

4. Let $R$ be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the $y$-axis, as shown in the figure above.
   (a) Find the area of $R$.
   (b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when $R$ is rotated about the horizontal line $y = 7$.
   (c) Region $R$ is the base of a solid. For each $y$, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the $y$-axis is a rectangle whose height is 3 times the length of its base in region $R$. Write, but do not evaluate, an integral expression that gives the volume of the solid.




9-B

4. Let $R$ be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.
   (a) Find the area of $R$.
   (b) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are squares. Find the volume of this solid.
   (c) Write, but do not evaluate, an integral expression for the volume of the solid generated when $R$ is rotated about the horizontal line $y = 2$.




4-B

6. Let $\ell$ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.

(a) Find $\int_0^1 x^n \, dx$ in terms of $n$.

(b) Let $T$ be the triangular region bounded by $\ell$, the $x$-axis, and the line $x = 1$. Show that the area of $T$ is $\frac{1}{2n}$.

(c) Let $S$ be the region bounded by the graph of $y = x^n$, the line $\ell$, and the $x$-axis. Express the area of $S$ in terms of $n$ and determine the value of $n$ that maximizes the area of $S$.

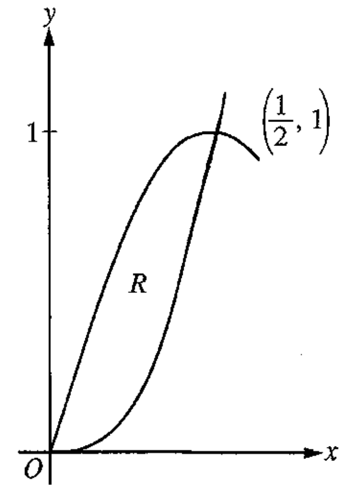
## Question 3

Let $R$ be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of $f$ at $x = \frac{1}{2}$.

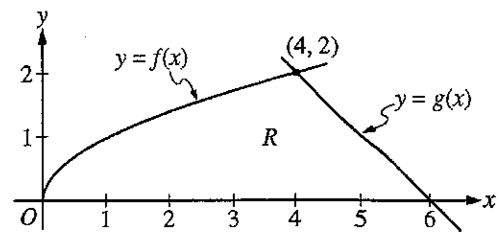
(b) Find the area of $R$.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when $R$ is rotated about the horizontal line $y = 1$.




| | |
|---|---|
| (a) $f\left(\frac{1}{2}\right) = 1$<br>$f'(x) = 24x^2$, so $f'\left(\frac{1}{2}\right) = 6$<br>An equation for the tangent line is $y = 1 + 6\left(x - \frac{1}{2}\right)$. | $2 : \begin{cases} 1 : f'\left(\frac{1}{2}\right) \\ 1 : \text{answer} \end{cases}$ |
| (b) Area $= \int_0^{1/2} (g(x) - f(x))\,dx$<br>$= \int_0^{1/2} \left(\sin(\pi x) - 8x^3\right) dx$<br>$= \left[-\frac{1}{\pi}\cos(\pi x) - 2x^4\right]_{x=0}^{x=1/2}$<br>$= -\frac{1}{8} + \frac{1}{\pi}$ | $4 : \begin{cases} 1 : \text{integrand} \\ 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$ |
| (c) $\pi\int_0^{1/2} \left((1 - f(x))^2 - (1 - g(x))^2\right) dx$<br>$= \pi\int_0^{1/2} \left(\left(1 - 8x^3\right)^2 - (1 - \sin(\pi x))^2\right) dx$ | $3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$ |

## Question 3




The functions $f$ and $g$ are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let $R$ be the region bounded by the $x$-axis and the graphs of $f$ and $g$, as shown in the figure above.

(a) Find the area of $R$.

(b) The region $R$ is the base of a solid. For each $y$, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the $y$-axis is a rectangle whose base lies in $R$ and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(c) There is a point $P$ on the graph of $f$ at which the line tangent to the graph of $f$ is perpendicular to the graph of $g$. Find the coordinates of point $P$.

---

(a) Area $= \int_0^4 \sqrt{x}\,dx + \frac{1}{2}\cdot 2\cdot 2 = \frac{2}{3}x^{3/2}\Big|_{x=0}^{x=4} + 2 = \frac{22}{3}$

$3: \begin{cases} 1: \text{integral} \\ 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

(b) $y = \sqrt{x} \;\Rightarrow\; x = y^2$

$y = 6 - x \;\Rightarrow\; x = 6 - y$

Width $= (6 - y) - y^2$

Volume $= \int_0^2 2y\left(6 - y - y^2\right) dy$

$3: \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

(c) $g'(x) = -1$

Thus a line perpendicular to the graph of $g$ has slope 1.

$f'(x) = \frac{1}{2\sqrt{x}}$

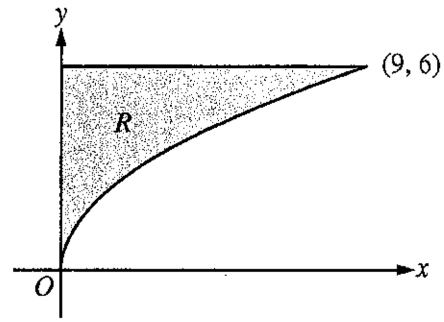
$\frac{1}{2\sqrt{x}} = 1 \;\Rightarrow\; x = \frac{1}{4}$

The point $P$ has coordinates $\left(\frac{1}{4}, \frac{1}{2}\right)$.

$3: \begin{cases} 1: f'(x) \\ 1: \text{equation} \\ 1: \text{answer} \end{cases}$

# AP® CALCULUS AB
# 2010 SCORING GUIDELINES

## Question 4




Let $R$ be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the $y$-axis, as shown in the figure above.

(a) Find the area of $R$.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when $R$ is rotated about the horizontal line $y = 7$.

(c) Region $R$ is the base of a solid. For each $y$, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the $y$-axis is a rectangle whose height is 3 times the length of its base in region $R$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

| | |
|---|---|
| (a) Area $= \int_0^9 \left(6 - 2\sqrt{x}\right) dx = \left(6x - \frac{4}{3}x^{3/2}\right)\Big\|_{x=0}^{x=9} = 18$ | $3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$ |
| (b) Volume $= \pi \int_0^9 \left(\left(7 - 2\sqrt{x}\right)^2 - (7-6)^2\right) dx$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$ |
| (c) Solving $y = 2\sqrt{x}$ for $x$ yields $x = \frac{y^2}{4}$. <br> Each rectangular cross section has area $\left(3\frac{y^2}{4}\right)\left(\frac{y^2}{4}\right) = \frac{3}{16}y^4$. <br> Volume $= \int_0^6 \frac{3}{16}y^4\, dy$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |



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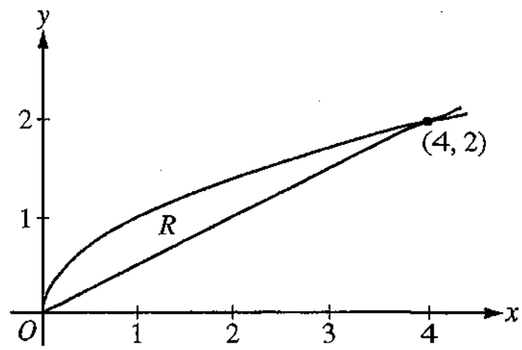
## Question 4

Let $R$ be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.




(a) Find the area of $R$.

(b) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are squares. Find the volume of this solid.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when $R$ is rotated about the horizontal line $y = 2$.

---

(a) Area $= \displaystyle\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{2}{3}x^{3/2} - \frac{x^2}{4}\Big|_{x=0}^{x=4} = \frac{4}{3}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \displaystyle\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx$

$= \displaystyle\frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12}\Big|_{x=0}^{x=4} = \frac{8}{15}$

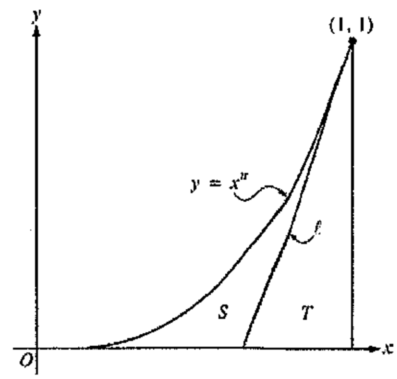
$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \displaystyle\int_0^4 \left(\left(2 - \frac{x}{2}\right)^2 - \left(2 - \sqrt{x}\right)^2\right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$

### Question 6

Let $\ell$ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.




(a) Find $\int_0^1 x^n\,dx$ in terms of $n$.

(b) Let $T$ be the triangular region bounded by $\ell$, the $x$-axis, and the line $x = 1$. Show that the area of $T$ is $\dfrac{1}{2n}$.

(c) Let $S$ be the region bounded by the graph of $y = x^n$, the line $\ell$, and the $x$-axis. Express the area of $S$ in terms of $n$ and determine the value of $n$ that maximizes the area of $S$.

---

(a) $\displaystyle\int_0^1 x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$

$$2 : \begin{cases} 1 : \text{antiderivative of } x^n \\ 1 : \text{answer} \end{cases}$$

(b) Let $b$ be the length of the base of triangle $T$.

$\dfrac{1}{b}$ is the slope of line $\ell$, which is $n$

$$\text{Area}(T) = \frac{1}{2}b(1) = \frac{1}{2n}$$

$$3 : \begin{cases} 1 : \text{slope of line } \ell \text{ is } n \\ 1 : \text{base of } T \text{ is } \dfrac{1}{n} \\ 1 : \text{shows area is } \dfrac{1}{2n} \end{cases}$$

(c)
$$\text{Area}(S) = \int_0^1 x^n\,dx - \text{Area}(T)$$
$$= \frac{1}{n+1} - \frac{1}{2n}$$

$$\frac{d}{dn}\text{Area}(S) = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = 0$$

$$2n^2 = (n+1)^2$$

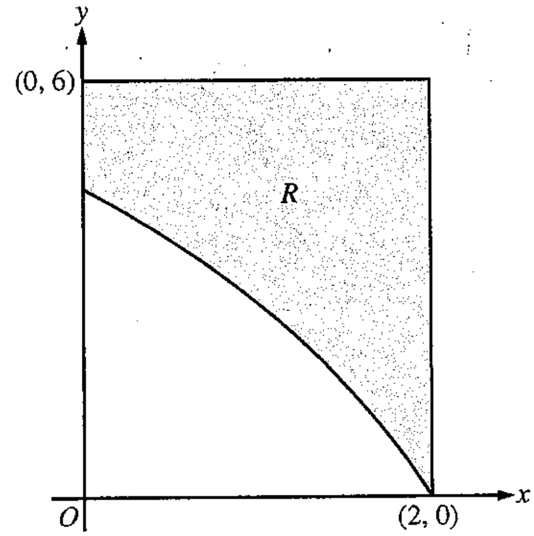
$$\sqrt{2}\,n = (n+1)$$

$$n = \frac{1}{\sqrt{2}-1} = 1 + \sqrt{2}$$

$$4 : \begin{cases} 1 : \text{area of } S \text{ in terms of } n \\ \cancel{1 : \text{derivative}} \\ 1 : \text{sets derivative equal to } 0 \\ 1 : \text{solves for } n \end{cases}$$

# Area between Curves / Volumes of Solid FRQ

with Calculator



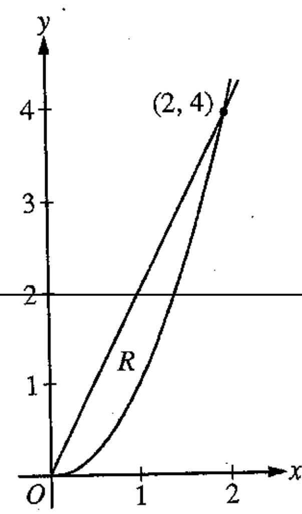

10B

1. In the figure above, $R$ is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

   (a) Find the area of $R$.

   (b) Find the volume of the solid generated when $R$ is revolved about the horizontal line $y = 8$.

   (c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of the solid.



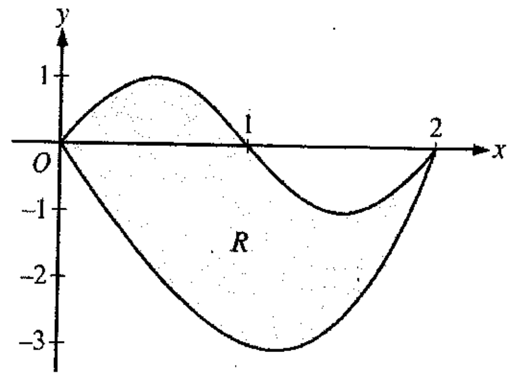

9-A

4. Let $R$ be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

   (a) Find the area of $R$.

   (b) The region $R$ is the base of a solid. For this solid, at each $x$ the cross section perpendicular to the $x$-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

   (c) Another solid has the same base $R$. For this solid, the cross sections perpendicular to the $y$-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.




8-A

1. Let $R$ be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.
   (a) Find the area of $R$.
   (b) The horizontal line $y = -2$ splits the region $R$ into two parts. Write, but do not evaluate, an integral expression for the area of the part of $R$ that is below this horizontal line.
   (c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of this solid.
   (d) The region $R$ models the surface of a small pond. At all points in $R$ at a distance $x$ from the $y$-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

8-B

1. Let $R$ be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$.
   (a) Find the area of $R$.
   (b) Find the volume of the solid generated when $R$ is rotated about the vertical line $x = -1$.
   (c) ~~The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $y$-axis are squares.~~ Find the volume of this solid.

7-A

1. Let $R$ be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1 + x^2}$ and below by the horizontal line $y = 2$.
   (a) Find the area of $R$.
   (b) Find the volume of the solid generated when $R$ is rotated about the $x$-axis.
   (c) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are semicircles. Find the volume of this solid.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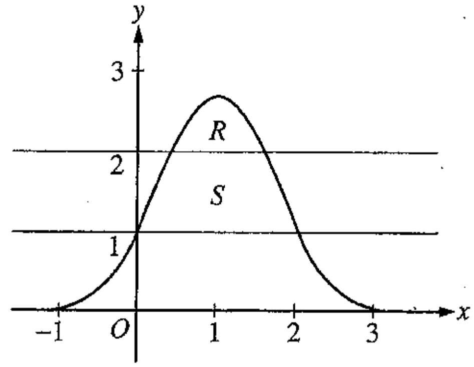


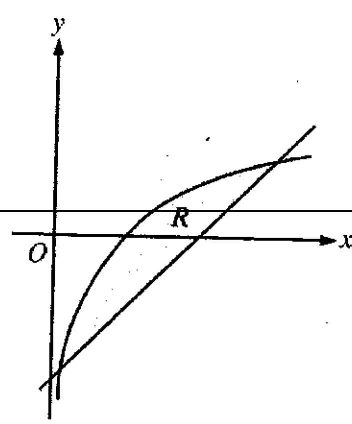

7-B

1. Let $R$ be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let $S$ be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

   (a) Find the area of $R$.

   (b) Find the area of $S$.

   (c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when $R$ is rotated about the horizontal line $y = 1$.




6-A

1. Let $R$ be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

   (a) Find the area of $R$.

   (b) Find the volume of the solid generated when $R$ is rotated about the horizontal line $y = -3$.

   (c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when $R$ is rotated about the $y$-axis.

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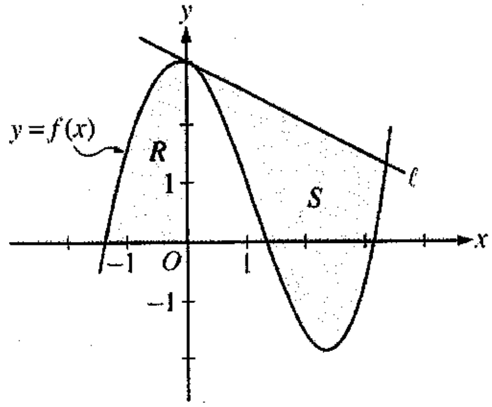


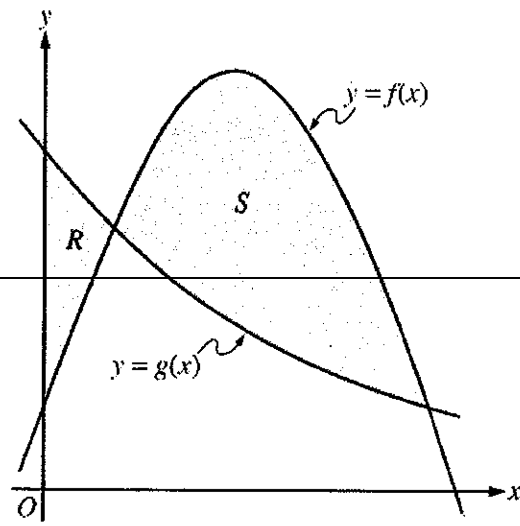

6-B

1. Let $f$ be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let $R$ be the shaded region in the second quadrant bounded by the graph of $f$, and let $S$ be the shaded region bounded by the graph of $f$ and line $\ell$, the line tangent to the graph of $f$ at $x = 0$, as shown above.

   (a) Find the area of $R$.

   (b) Find the volume of the solid generated when $R$ is rotated about the horizontal line $y = -2$.

   (c) Write, but do not evaluate, an integral expression that can be used to find the area of $S$.



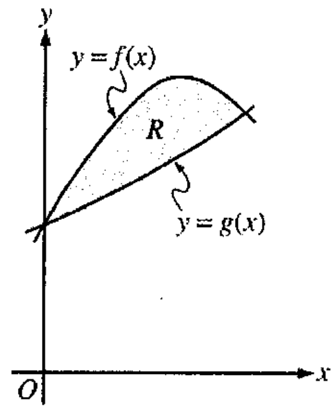

5-A

1. Let $f$ and $g$ be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let $R$ be the shaded region in the first quadrant enclosed by the $y$-axis and the graphs of $f$ and $g$, and let $S$ be the shaded region in the first quadrant enclosed by the graphs of $f$ and $g$, as shown in the figure above.

   (a) Find the area of $R$.

   (b) Find the area of $S$.

   (c) Find the volume of the solid generated when $S$ is revolved about the horizontal line $y = -1$.



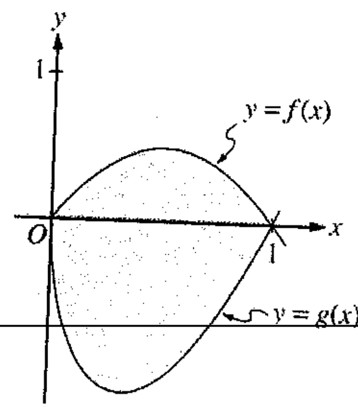

5-B

1. Let $f$ and $g$ be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let $R$ be the shaded region in the first quadrant enclosed by the graphs of $f$ and $g$ as shown in the figure above.

   (a) Find the area of $R$.

   (b) Find the volume of the solid generated when $R$ is revolved about the $x$-axis.

   (c) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are semicircles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of this solid.




4-A

2. Let $f$ and $g$ be the functions given by $f(x) = 2x(1 - x)$ and $g(x) = 3(x - 1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of $f$ and $g$ are shown in the figure above.

   (a) Find the area of the shaded region enclosed by the graphs of $f$ and $g$.

   (b) Find the volume of the solid generated when the shaded region enclosed by the graphs of $f$ and $g$ is revolved about the horizontal line $y = 2$.

   (c) Let $h$ be the function given by $h(x) = kx(1 - x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of $h$ and $g$ is the base of a solid with square cross sections perpendicular to the $x$-axis. There is a value of $k$ for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of $k$.



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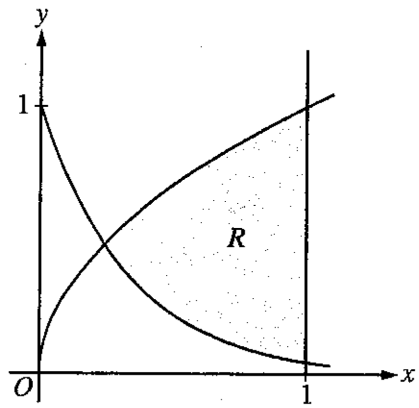
4-B

1. Let $R$ be the region enclosed by the graph of $y = \sqrt{x-1}$, the vertical line $x = 10$, and the $x$-axis.

   (a) Find the area of $R$.

   (b) Find the volume of the solid generated when $R$ is revolved about the horizontal line $y = 3$.

   (c) Find the volume of the solid generated when $R$ is revolved about the vertical line $x = 10$.



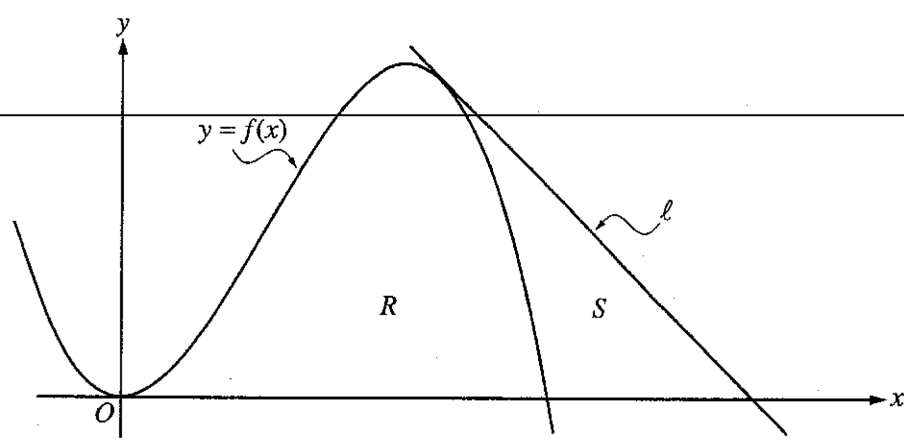

3-A

1. Let $R$ be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.

   (a) Find the area of $R$.

   (b) Find the volume of the solid generated when $R$ is revolved about the horizontal line $y = 1$.

   (c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a rectangle whose height is 5 times the length of its base in region $R$. Find the volume of this solid.




3-B

1. Let $f$ be the function given by $f(x) = 4x^2 - x^3$, and let $\ell$ be the line $y = 18 - 3x$, where $\ell$ is tangent to the graph of $f$. Let $R$ be the region bounded by the graph of $f$ and the $x$-axis, and let $S$ be the region bounded by the graph of $f$, the line $\ell$, and the $x$-axis, as shown above.

   (a) Show that $\ell$ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

   (b) Find the area of $S$.

   (c) Find the volume of the solid generated when $R$ is revolved about the $x$-axis.

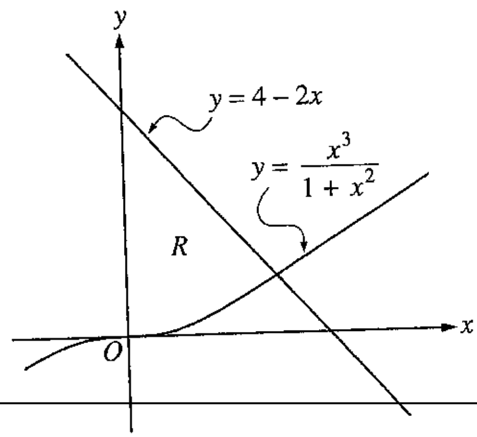
**2-A**

1. Let $f$ and $g$ be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

   (a) Find the area of the region enclosed by the graphs of $f$ and $g$ between $x = \frac{1}{2}$ and $x = 1$.

   (b) Find the volume of the solid generated when the region enclosed by the graphs of $f$ and $g$ between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

   (c) Let $h$ be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.



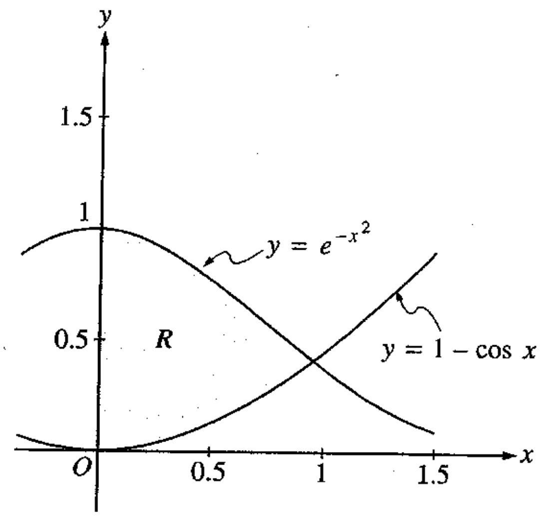

**2B**

1. Let $R$ be the region bounded by the $y$-axis and the graphs of $y = \dfrac{x^3}{1+x^2}$ and $y = 4 - 2x$, as shown in the figure above.

   (a) Find the area of $R$.

   (b) Find the volume of the solid generated when $R$ is revolved about the $x$-axis.

   (c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of this solid.




00

1. Let $R$ be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the $y$-axis, as shown in the figure above.
   (a) Find the area of the region $R$.
   (b) Find the volume of the solid generated when the region $R$ is revolved about the $x$-axis.
   (c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of this solid.

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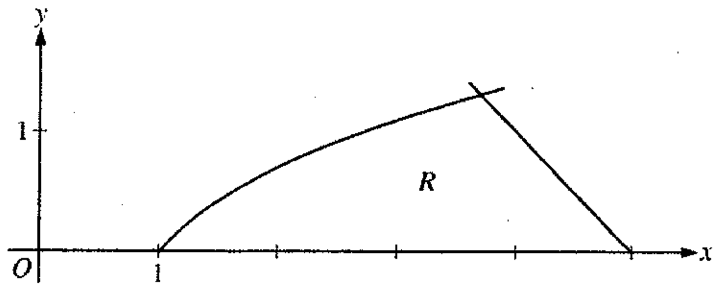
1. Let $R$ be the region bounded by the $x$–axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.
   (a) Find the area of the region $R$.
   (b) Find the value of $h$ such that the vertical line $x = h$ divides the region $R$ into two regions of equal area.
   (c) Find the volume of the solid generated when $R$ is revolved about the $x$–axis.
   (d) The vertical line $x = k$ divides the region $R$ into two regions such that when these two regions are revolved about the $x$–axis, they generate solids with equal volumes. Find the value of $k$.




12-A 2. Let $R$ be the region in the first quadrant bounded by the $x$-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure above.

(a) Find the area of $R$.

(b) Region $R$ is the base of a solid. For the solid, each cross section perpendicular to the $x$-axis is a square. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(c) The horizontal line $y = k$ divides $R$ into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of $k$.

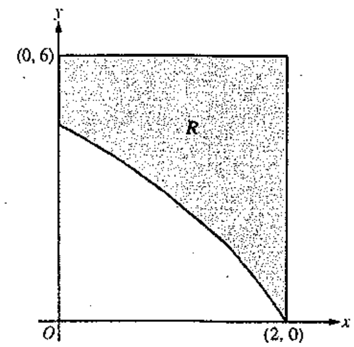
## Question 1

In the figure above, $R$ is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3-x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is revolved about the horizontal line $y = 8$.

(c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of the solid.

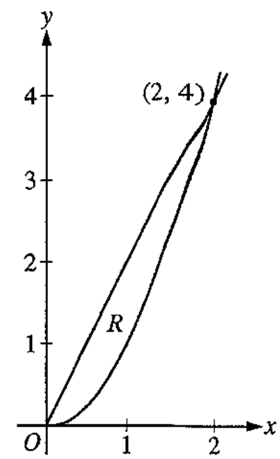



| | |
|---|---|
| | 1 : Correct limits in an integral in (a), (b), or (c) |
| (a) $\int_0^2 (6 - 4\ln(3-x))\,dx = 6.816$ or $6.817$ | $2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (b) $\pi\int_0^2 \left((8 - 4\ln(3-x))^2 - (8-6)^2\right)dx$ $= 168.179$ or $168.180$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (c) $\int_0^2 (6 - 4\ln(3-x))^2\,dx = 26.266$ or $26.267$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |

# AP® CALCULUS AB
# 2009 SCORING GUIDELINES

## Question 4

Let $R$ be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.




(a) Find the area of $R$.

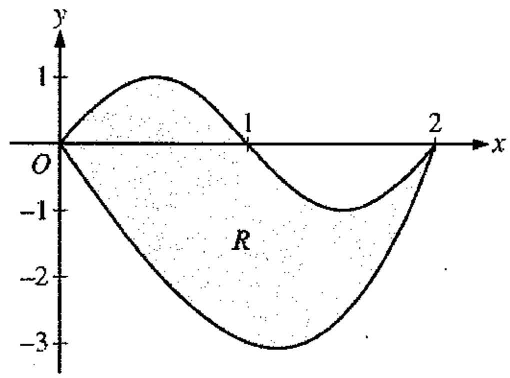
(b) The region $R$ is the base of a solid. For this solid, at each $x$ the cross section perpendicular to the $x$-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

(c) Another solid has the same base $R$. For this solid, the cross sections perpendicular to the $y$-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

| | |
|---|---|
| (a) Area $= \int_0^2 \left(2x - x^2\right) dx$ <br> $= x^2 - \frac{1}{3}x^3 \Big\vert_{x=0}^{x=2}$ <br> $= \frac{4}{3}$ | $3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$ |
| (b) Volume $= \int_0^2 \sin\left(\frac{\pi}{2}x\right) dx$ <br> $= -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big\vert_{x=0}^{x=2}$ <br> $= \frac{4}{\pi}$ | $3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$ |
| (c) Volume $= \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)^2 dy$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \end{cases}$ |

## AP® CALCULUS AB
## 2008 SCORING GUIDELINES

### Question 1




Let $R$ be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of $R$.

(b) The horizontal line $y = -2$ splits the region $R$ into two parts. Write, but do not evaluate, an integral expression for the area of the part of $R$ that is below this horizontal line.

(c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of this solid.

(d) The region $R$ models the surface of a small pond. At all points in $R$ at a distance $x$ from the $y$-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

| | |
|---|---|
| (a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$<br>Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$ | $3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$<br>The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$ | $2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \end{cases}$ |
| (c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$ | $2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or $8.370$ | $2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |

## Question 1

Let $R$ be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$.

(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is rotated about the vertical line $x = -1$.

(c) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $y$-axis are squares. Find the volume of this solid.

---

The graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$ intersect at the points $(0, 0)$ and $(9, 3)$.

(a) $\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx = 4.5$

OR

$\int_0^3 \left(3y - y^2\right) dy = 4.5$

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $\pi \int_0^3 \left((3y+1)^2 - \left(y^2+1\right)^2\right) dy$

$= \frac{207\pi}{5} = 130.061 \text{ or } 130.062$

$4 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^3 \left(3y - y^2\right)^2 dy = 8.1$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

## Question 1

Let $R$ be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of $R$.

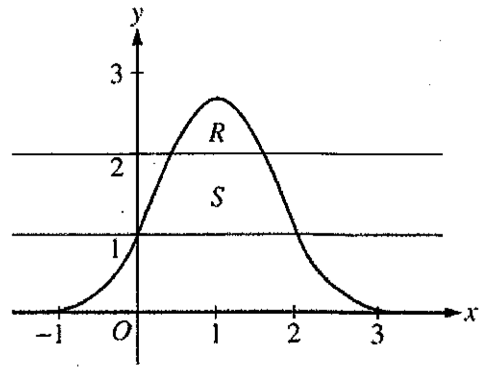
(b) Find the volume of the solid generated when $R$ is rotated about the $x$-axis.

(c) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are semicircles. Find the volume of this solid.

| | |
|---|---|
| $\dfrac{20}{1+x^2} = 2$ when $x = \pm 3$ | 1 : correct limits in an integral in (a), (b), or (c) |
| (a) Area $= \displaystyle\int_{-3}^{3} \left( \frac{20}{1+x^2} - 2 \right) dx = 37.961$ or $37.962$ | $2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (b) Volume $= \pi \displaystyle\int_{-3}^{3} \left( \left( \frac{20}{1+x^2} \right)^2 - 2^2 \right) dx = 1871.190$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (c) Volume $= \dfrac{\pi}{2} \displaystyle\int_{-3}^{3} \left( \frac{1}{2} \left( \frac{20}{1+x^2} - 2 \right) \right)^2 dx$ $= \dfrac{\pi}{8} \displaystyle\int_{-3}^{3} \left( \frac{20}{1+x^2} - 2 \right)^2 dx = 174.268$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |

### Question 1

Let $R$ be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let $S$ be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.




(a) Find the area of $R$.

(b) Find the area of $S$.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when $R$ is rotated about the horizontal line $y = 1$.

---

| | |
|---|---|
| $e^{2x-x^2} = 2$ when $x = 0.446057, 1.553943$<br>Let $P = 0.446057$ and $Q = 1.553943$ | |
| (a) Area of $R = \int_P^Q \left(e^{2x-x^2} - 2\right) dx = 0.514$ | $3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$ |
| (b) $e^{2x-x^2} = 1$ when $x = 0,\ 2$<br>Area of $S = \int_0^2 \left(e^{2x-x^2} - 1\right) dx - \text{Area of } R$<br>$= 2.06016 - \text{Area of } R = 1.546$ | $3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$ |
| OR<br>$\int_0^P \left(e^{2x-x^2} - 1\right) dx + (Q - P)\cdot 1 + \int_Q^2 \left(e^{2x-x^2} - 1\right) dx$<br>$= 0.219064 + 1.107886 + 0.219064 = 1.546$ | |
| (c) Volume $= \pi \int_P^Q \left(\left(e^{2x-x^2} - 1\right)^2 - (2-1)^2\right) dx$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{constant and limits} \end{cases}$ |

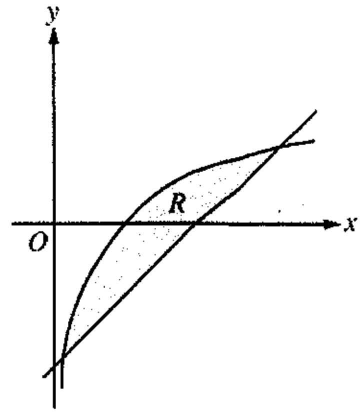
## Question 1

Let $R$ be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when $R$ is rotated about the $y$-axis.




| | |
|---|---|
| $\ln(x) = x - 2$ when $x = 0.15859$ and $3.14619$.<br>Let $S = 0.15859$ and $T = 3.14619$ | |
| (a) Area of $R = \int_S^T (\ln(x) - (x-2))\,dx = 1.949$ | $3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$ |
| (b) Volume $= \pi \int_S^T \left( (\ln(x)+3)^2 - (x-2+3)^2 \right) dx$<br>$= 34.198$ or $34.199$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits, constant, and answer} \end{cases}$ |
| (c) Volume $= \pi \int_{S-2}^{T-2} \left( (y+2)^2 - \left(e^y\right)^2 \right) dy$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$ |





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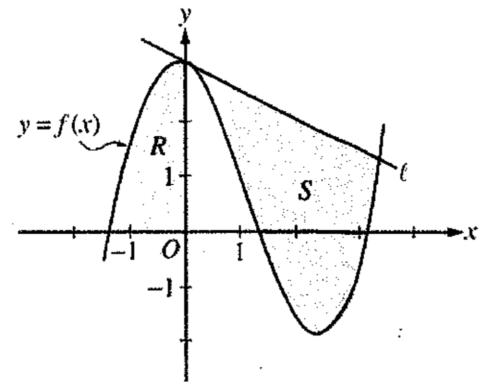
# AP® CALCULUS AB
# 2006 SCORING GUIDELINES (Form B)

## Question 1

Let $f$ be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let $R$ be the shaded region in the second quadrant bounded by the graph of $f$, and let $S$ be the shaded region bounded by the graph of $f$ and line $\ell$, the line tangent to the graph of $f$ at $x = 0$, as shown above.




(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an integral expression that can be used to find the area of $S$.

---

For $x < 0$, $f(x) = 0$ when $x = -1.37312$.
Let $P = -1.37312$.

(a) Area of $R = \int_P^0 f(x)\,dx = 2.903$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_P^0 \left( (f(x) + 2)^2 - 4 \right) dx = 59.361$

$4 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

---

(c) The equation of the tangent line $\ell$ is $y = 3 - \frac{1}{2}x$.

The graph of $f$ and line $\ell$ intersect at $A = 3.38987$.

Area of $S = \int_0^A \left( \left(3 - \frac{1}{2}x\right) - f(x) \right) dx$

$3 : \begin{cases} 1 : \text{tangent line} \\ 1 : \text{integrand} \\ 1 : \text{limits} \end{cases}$





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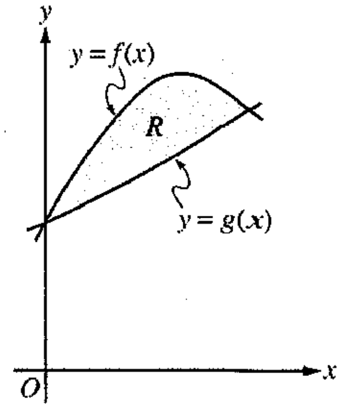
## Question 1

Let $f$ and $g$ be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let $R$ be the shaded region in the first quadrant enclosed by the graphs of $f$ and $g$ as shown in the figure above.




(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is revolved about the $x$-axis.

(c) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are semicircles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of this solid.

---

| | |
|---|---|
| The graphs of $f$ and $g$ intersect in the first quadrant at $(S, T) = (1.13569, 1.76446)$. | 1 : correct limits in an integral in (a), (b), or (c) |
| (a) Area $= \int_0^S (f(x) - g(x))\,dx$ <br> $= \int_0^S \left(1 + \sin(2x) - e^{x/2}\right) dx$ <br> $= 0.429$ | $2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (b) Volume $= \pi \int_0^S \left((f(x))^2 - (g(x))^2\right) dx$ <br> $= \pi \int_0^S \left((1 + \sin(2x))^2 - \left(e^{x/2}\right)^2\right) dx$ <br> $= 4.266$ or $4.267$ | $3 : \begin{cases} 2 : \text{integrand} \\ \quad \langle -1 \rangle \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if integral not of form} \\ \qquad c\int_a^b \left(R^2(x) - r^2(x)\right) dx \\ 1 : \text{answer} \end{cases}$ |
| (c) Volume $= \int_0^S \frac{\pi}{2}\left(\frac{f(x) - g(x)}{2}\right)^2 dx$ <br> $= \int_0^S \frac{\pi}{2}\left(\frac{1 + \sin(2x) - e^{x/2}}{2}\right)^2 dx$ <br> $= 0.077$ or $0.078$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |





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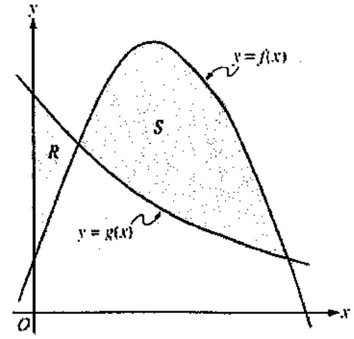
**Question 1**

Let $f$ and $g$ be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let $R$ be the shaded region in the first quadrant enclosed by the $y$-axis and the graphs of $f$ and $g$, and let $S$ be the shaded region in the first quadrant enclosed by the graphs of $f$ and $g$, as shown in the figure above.




(a) Find the area of $R$.

(b) Find the area of $S$.

(c) Find the volume of the solid generated when $S$ is revolved about the horizontal line $y = -1$.

---

$f(x) = g(x)$ when $\frac{1}{4} + \sin(\pi x) = 4^{-x}$.

$f$ and $g$ intersect when $x = 0.178218$ and when $x = 1$.
Let $a = 0.178218$.

| | |
|---|---|
| (a) $\int_0^a (g(x) - f(x))\,dx = 0.064$ or $0.065$ | $3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (b) $\int_a^1 (f(x) - g(x))\,dx = 0.410$ | $3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (c) $\pi \int_a^1 \left((f(x)+1)^2 - (g(x)+1)^2\right) dx = 4.558$ or $4.559$ | $3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits, constant, and answer} \end{cases}$ |





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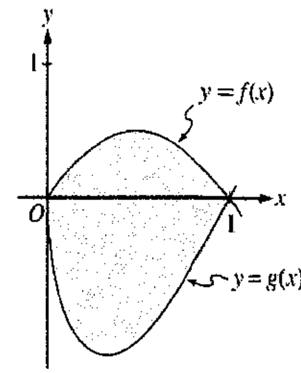
### Question 2




Let $f$ and $g$ be the functions given by $f(x) = 2x(1-x)$ and $g(x) = 3(x-1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of $f$ and $g$ are shown in the figure above.

(a) Find the area of the shaded region enclosed by the graphs of $f$ and $g$.

(b) Find the volume of the solid generated when the shaded region enclosed by the graphs of $f$ and $g$ is revolved about the horizontal line $y = 2$.

(c) Let $h$ be the function given by $h(x) = kx(1-x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of $h$ and $g$ is the base of a solid with square cross sections perpendicular to the $x$-axis. There is a value of $k$ for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of $k$.

---

(a) Area $= \int_0^1 (f(x) - g(x))\,dx$

$= \int_0^1 \left(2x(1-x) - 3(x-1)\sqrt{x}\right) dx = 1.133$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^1 \left((2 - g(x))^2 - (2 - f(x))^2\right) dx$

$= \pi \int_0^1 \left((2 - 3(x-1)\sqrt{x})^2 - (2 - 2x(1-x))^2\right) dx$

$= 16.179$

$4 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \\ \quad \langle -1 \rangle \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if integral not of form} \\ \qquad c\int_a^b \left(R^2(x) - r^2(x)\right) dx \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \int_0^1 (h(x) - g(x))^2\,dx$

$\int_0^1 \left(kx(1-x) - 3(x-1)\sqrt{x}\right)^2 dx = 15$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$



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## Question 1

Let $R$ be the region enclosed by the graph of $y = \sqrt{x-1}$, the vertical line $x = 10$, and the $x$-axis.

(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is revolved about the horizontal line $y = 3$.

(c) Find the volume of the solid generated when $R$ is revolved about the vertical line $x = 10$.

| | |
|---|---|
| (a) Area $= \int_1^{10} \sqrt{x-1}\,dx = 18$ | $3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$ |
| (b) Volume $= \pi \int_1^{10} \left(9 - \left(3 - \sqrt{x-1}\right)^2\right) dx$ $= 212.057$ or $212.058$ | $3: \begin{cases} 1: \text{limits and constant} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$ |
| (c) Volume $= \pi \int_0^3 \left(10 - \left(y^2+1\right)\right)^2 dy$ $= 407.150$ | $3: \begin{cases} 1: \text{limits and constant} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$ |



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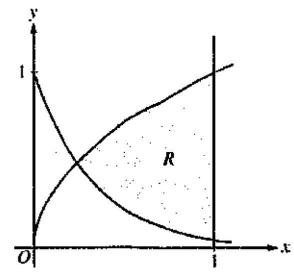
### Question 1

Let $R$ be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.




(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is revolved about the horizontal line $y = 1$.

(c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a rectangle whose height is 5 times the length of its base in region $R$. Find the volume of this solid.

| | |
|---|---|
| Point of intersection<br>$e^{-3x} = \sqrt{x}$ at $(T, S) = (0.238734, 0.488604)$ | 1: Correct limits in an integral in (a), (b), or (c) |
| (a) Area $= \int_T^1 \left(\sqrt{x} - e^{-3x}\right) dx$<br>$= 0.442$ or $0.443$ | $2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$ |
| (b) Volume $= \pi \int_T^1 \left(\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2\right) dx$<br>$= 0.453\pi$ or $1.423$ or $1.424$ | $3: \begin{cases} 2: \text{integrand} \\ \quad <-1> \text{ reversal} \\ \quad <-1> \text{ error with constant} \\ \quad <-1> \text{ omits 1 in one radius} \\ \quad <-2> \text{ other errors} \\ 1: \text{answer} \end{cases}$ |
| (c) Length $= \sqrt{x} - e^{-3x}$<br>Height $= 5\left(\sqrt{x} - e^{-3x}\right)$<br><br>Volume $= \int_T^1 5\left(\sqrt{x} - e^{-3x}\right)^2 dx = 1.554$ | $3: \begin{cases} 2: \text{integrand} \\ \quad <-1> \text{ incorrect but has } \sqrt{x} - e^{-3x} \text{ as a factor} \\ 1: \text{answer} \end{cases}$ |





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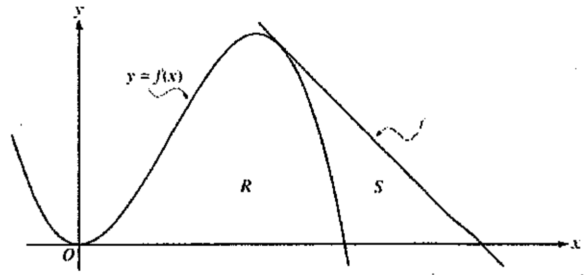
# AP® CALCULUS AB
# 2003 SCORING GUIDELINES (Form B)

## Question 1

Let $f$ be the function given by $f(x) = 4x^2 - x^3$, and let $\ell$ be the line $y = 18 - 3x$, where $\ell$ is tangent to the graph of $f$. Let $R$ be the region bounded by the graph of $f$ and the $x$-axis, and let $S$ be the region bounded by the graph of $f$, the line $\ell$, and the $x$-axis, as shown above.




(a) Show that $\ell$ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of $S$.

(c) Find the volume of the solid generated when $R$ is revolved about the $x$-axis.

---

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$

$f(3) = 36 - 27 = 9$

Tangent line at $x = 3$ is

$y = -3(x - 3) + 9 = -3x + 18$,

which is the equation of line $\ell$.

$$2 : \begin{cases} 1 : \text{finds } f'(3) \text{ and } f(3) \\ 1 : \begin{cases} \text{finds equation of tangent line} \\ \text{or} \\ \text{shows } (3,9) \text{ is on both the} \\ \text{graph of } f \text{ and line } \ell \end{cases} \end{cases}$$

(b) $f(x) = 0$ at $x = 4$

The line intersects the $x$-axis at $x = 6$.

$$\text{Area} = \frac{1}{2}(3)(9) - \int_3^4 \left(4x^2 - x^3\right) dx$$
$$= 7.916 \text{ or } 7.917$$

OR

$$\text{Area} = \int_3^4 \left((18 - 3x) - \left(4x^2 - x^3\right)\right) dx + \frac{1}{2}(2)(18 - 12)$$
$$= 7.916 \text{ or } 7.917$$

$$4 : \begin{cases} 2 : \text{integral for non-triangular region} \\ \quad 1 : \text{limits} \\ \quad 1 : \text{integrand} \\ 1 : \text{area of triangular region} \\ 1 : \text{answer} \end{cases}$$

(c) $$\text{Volume} = \pi \int_0^4 \left(4x^2 - x^3\right)^2 dx$$
$$= 156.038\pi \text{ or } 490.208$$

$$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$





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## Question 1

Let $f$ and $g$ be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

(a) Find the area of the region enclosed by the graphs of $f$ and $g$ between $x = \frac{1}{2}$ and $x = 1$.

(b) Find the volume of the solid generated when the region enclosed by the graphs of $f$ and $g$ between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

(c) Let $h$ be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.

---

(a) Area $= \int_{1/2}^{1} (e^x - \ln x)\,dx = 1.222$ or $1.223$

$$2 \begin{cases} 1: \text{ integral} \\ 1: \text{ answer} \end{cases}$$

(b) Volume $= \pi \int_{1/2}^{1} \left( (4 - \ln x)^2 - (4 - e^x)^2 \right) dx$

$= 7.515\pi$ or $23.609$

$$4 \begin{cases} 1: \text{ limits and constant} \\ 2: \text{ integrand} \\ \quad < -1 > \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if not of the form} \\ \quad k\int_a^b \left( R(x)^2 - r(x)^2 \right) dx \\ 1: \text{ answer} \end{cases}$$

(c) $h'(x) = f'(x) - g'(x) = e^x - \frac{1}{x} = 0$

$x = 0.567143$

Absolute minimum value and absolute maximum value occur at the critical point or at the endpoints.

$h(0.567143) = 2.330$

$h(0.5) = 2.3418$

$h(1) = 2.718$

The absolute minimum is 2.330.

The absolute maximum is 2.718.

$$3 \begin{cases} 1: \text{ considers } h'(x) = 0 \\ 1: \text{ identifies critical point} \\ \quad \text{and endpoints as candidates} \\ 1: \text{ answers} \end{cases}$$

Note: Errors in computation come off the third point.





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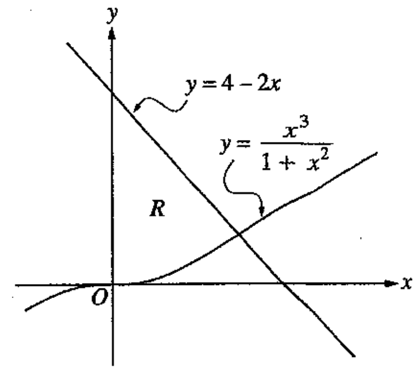
### Question 1

Let $R$ be the region bounded by the $y$-axis and the graphs of $y = \dfrac{x^3}{1+x^2}$ and $y = 4 - 2x$, as shown in the figure above.




(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is revolved about the $x$-axis.

(c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of this solid.

| | |
|---|---|
| Region $R$<br>$\dfrac{x^3}{1+x^2} = 4 - 2x$ at $x = 1.487664 = A$ | 1 : Correct limits in an integral in (a), (b), or (c). |
| (a) Area $= \displaystyle\int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right) dx$<br>$= 3.214$ or $3.215$ | 2 $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$ |
| (b) Volume<br>$= \pi \displaystyle\int_0^A \left((4-2x)^2 - \left(\frac{x^3}{1+x^2}\right)^2\right) dx$<br>$= 31.884$ or $31.885$ or $10.149\pi$ | 3 $\begin{cases} 2 : \text{integrand and constant} \\ \quad <-1> \text{ each error} \\ 1 : \text{answer} \end{cases}$ |
| (c) Volume $= \displaystyle\int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right)^2 dx$<br>$= 8.997$ | 3 $\begin{cases} 2 : \text{integrand} \\ \quad <-1> \text{ each error} \\ \quad \text{note: } 0/2 \text{ if not of the form} \\ \qquad k\displaystyle\int_c^d (f(x) - g(x))^2\, dx \\ 1 : \text{answer} \end{cases}$ |





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# Assorted Miscellaneous Derivatives/Integral FRQs

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 6 | 4 | 2 | 5 |
| 2 | 9 | 2 | 3 | 1 |
| 3 | 10 | −4 | 4 | 2 |
| 4 | −1 | 3 | 6 | 7 |

7-A

3. The functions $f$ and $g$ are differentiable for all real numbers, and $g$ is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of $x$. The function $h$ is given by $h(x) = f(g(x)) - 6$.

   (a) Explain why there must be a value $r$ for $1 < r < 3$ such that $h(r) = -5$.

   (b) Explain why there must be a value $c$ for $1 < c < 3$ such that $h'(c) = -5$.

   (c) Let $w$ be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

   (d) If $g^{-1}$ is the inverse function of $g$, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

| $x$ | −1.5 | −1.0 | −0.5 | 0 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −1 | −4 | −6 | −7 | −6 | −4 | −1 |
| $f'(x)$ | −7 | −5 | −3 | 0 | 3 | 5 | 7 |

2-A

6. Let $f$ be a function that is differentiable for all real numbers. The table above gives the values of $f$ and its derivative $f'$ for selected points $x$ in the closed interval $-1.5 \le x \le 1.5$. The second derivative of $f$ has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

   (a) Evaluate $\int_0^{1.5} (3f'(x) + 4)\,dx$. Show the work that leads to your answer.

   (b) Write an equation of the line tangent to the graph of $f$ at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

   (c) Find a positive real number $r$ having the property that there must exist a value $c$ with $0 < c < 0.5$ and $f''(c) = r$. Give a reason for your answer.

   (d) Let $g$ be the function given by $g(x) = \begin{cases} 2x^2 - x - 7 & \text{for } x < 0 \\ 2x^2 + x - 7 & \text{for } x \ge 0. \end{cases}$

   The graph of $g$ passes through each of the points $(x, f(x))$ given in the table above. Is it possible that $f$ and $g$ are the same function? Give a reason for your answer.

8-B

4. The functions $f$ and $g$ are given by $f(x) = \int_0^{3x} \sqrt{4+t^2}\, dt$ and $g(x) = f(\sin x)$.

(a) Find $f'(x)$ and $g'(x)$.

(b) Write an equation for the line tangent to the graph of $y = g(x)$ at $x = \pi$.

(c) Write, but do not evaluate, an integral expression that represents the maximum value of $g$ on the interval $0 \le x \le \pi$. Justify your answer.

9-B

6. Let $f$ be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let $g$ be the function given by $g(x) = f(f(x))$.

(a) Explain why there must be a value $c$ for $2 < c < 5$ such that $f'(c) = -1$.

(b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value $k$ for $2 < k < 5$ such that $g''(k) = 0$.

(c) Show that if $f''(x) = 0$ for all $x$, then the graph of $g$ does not have a point of inflection.

(d) Let $h(x) = f(x) - x$. Explain why there must be a value $r$ for $2 < r < 5$ such that $h(r) = 0$.

6-A

6. The twice-differentiable function $f$ is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2,\quad f'(0) = -4,\ \text{and}\ f''(0) = 3.$$

(a) The function $g$ is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where $a$ is a constant. Find $g'(0)$ and $g''(0)$ in terms of $a$. Show the work that leads to your answers.

(b) The function $h$ is given by $h(x) = \cos(kx)f(x)$ for all real numbers, where $k$ is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of $h$ at $x = 0$.


6-B

3. The figure above is the graph of a function of $x$, which models the height of a skateboard ramp. The function meets the following requirements.

   (i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.

   (ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.

   (iii) Between $x = 0$ and $x = 4$, the function is increasing.

(a) Let $f(x) = ax^2$, where $a$ is a nonzero constant. Show that it is not possible to find a value for $a$ so that $f$ meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where $c$ is a nonzero constant. Find the value of $c$ so that $g$ meets requirement (ii) above. Show the work that leads to your answer.

(c) Using the function $g$ and your value of $c$ from part (b), show that $g$ does not meet requirement (iii) above.

(d) Let $h(x) = \frac{x^n}{k}$, where $k$ is a nonzero constant and $n$ is a positive integer. Find the values of $k$ and $n$ so that $h$ meets requirement (ii) above. Show that $h$ also meets requirements (i) and (iii) above.

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12-A

4. The function $f$ is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

   (a) Find $f'(x)$.

   (b) Write an equation for the line tangent to the graph of $f$ at $x = -3$.

   (c) Let $g$ be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

   Is $g$ continuous at $x = -3$? Use the definition of continuity to explain your answer.

   (d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\,dx$.



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## Question 3

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 6 | 4 | 2 | 5 |
| 2 | 9 | 2 | 3 | 1 |
| 3 | 10 | $-4$ | 4 | 2 |
| 4 | $-1$ | 3 | 6 | 7 |

The functions $f$ and $g$ are differentiable for all real numbers, and $g$ is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of $x$. The function $h$ is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value $r$ for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value $c$ for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let $w$ be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If $g^{-1}$ is the inverse function of $g$, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

---

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$

$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$

Since $h(3) < -5 < h(1)$ and $h$ is continuous, by the Intermediate Value Theorem, there exists a value $r$, $1 < r < 3$, such that $h(r) = -5$.

$2: \begin{cases} 1: h(1) \text{ and } h(3) \\ 1: \text{conclusion, using IVT} \end{cases}$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$

Since $h$ is continuous and differentiable, by the Mean Value Theorem, there exists a value $c$, $1 < c < 3$, such that $h'(c) = -5$.

$2: \begin{cases} 1: \dfrac{h(3) - h(1)}{3 - 1} \\ 1: \text{conclusion, using MVT} \end{cases}$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

$2: \begin{cases} 1: \text{apply chain rule} \\ 1: \text{answer} \end{cases}$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.

$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$

An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

$3: \begin{cases} 1: g^{-1}(2) \\ 1: \left(g^{-1}\right)'(2) \\ 1: \text{tangent line equation} \end{cases}$





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## Question 6

| $x$ | $-1.5$ | $-1.0$ | $-0.5$ | $0$ | $0.5$ | $1.0$ | $1.5$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-1$ | $-4$ | $-6$ | $-7$ | $-6$ | $-4$ | $-1$ |
| $f'(x)$ | $-7$ | $-5$ | $-3$ | $0$ | $3$ | $5$ | $7$ |

Let $f$ be a function that is differentiable for all real numbers. The table above gives the values of $f$ and its derivative $f'$ for selected points $x$ in the closed interval $-1.5 \le x \le 1.5$. The second derivative of $f$ has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

(a) Evaluate $\int_0^{1.5} (3f'(x) + 4)\,dx$. Show the work that leads to your answer.

(b) Write an equation of the line tangent to the graph of $f$ at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

(c) Find a positive real number $r$ having the property that there must exist a value $c$ with $0 < c < 0.5$ and $f''(c) = r$. Give a reason for your answer.

(d) Let $g$ be the function given by $g(x) = \begin{cases} 2x^2 - x - 7 & \text{for } x < 0 \\ 2x^2 + x - 7 & \text{for } x \ge 0. \end{cases}$

The graph of $g$ passes through each of the points $(x, f(x))$ given in the table above. Is it possible that $f$ and $g$ are the same function? Give a reason for your answer.

---

(a) $\int_0^{1.5} (3f'(x) + 4)\,dx = 3\int_0^{1.5} f'(x)\,dx + \int_0^{1.5} 4\,dx$

$= 3f(x) + 4x\Big|_0^{1.5} = 3(-1 - (-7)) + 4(1.5) = 24$

$2 \begin{cases} 1: \text{ antiderivative} \\ 1: \text{ answer} \end{cases}$

(b) $y = 5(x - 1) - 4$

$f(1.2) \approx 5(0.2) - 4 = -3$

The approximation is less than $f(1.2)$ because the graph of $f$ is concave up on the interval $1 < x < 1.2$.

$3 \begin{cases} 1: \text{ tangent line} \\ 1: \text{ computes } y \text{ on tangent line at } x = 1.2 \\ 1: \text{ answer with reason} \end{cases}$

(c) By the Mean Value Theorem there is a $c$ with $0 < c < 0.5$ such that

$$f''(c) = \frac{f'(0.5) - f'(0)}{0.5 - 0} = \frac{3 - 0}{0.5} = 6 = r$$

$2 \begin{cases} 1: \text{ reference to MVT for } f' \text{ (or differentiability of } f'\text{)} \\ 1: \text{ value of } r \text{ for interval } 0 \le x \le 0.5 \end{cases}$

(d) $\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} (4x - 1) = -1$

$\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} (4x + 1) = +1$

Thus $g'$ is not continuous at $x = 0$, but $f'$ is continuous at $x = 0$, so $f \ne g$.

OR

$g''(x) = 4$ for all $x \ne 0$, but it was shown in part (c) that $f''(c) = 6$ for some $c \ne 0$, so $f \ne g$.

$2 \begin{cases} 1: \text{ answers "no" with reference to } g' \text{ or } g'' \\ 1: \text{ correct reason} \end{cases}$





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## Question 4

The functions $f$ and $g$ are given by $f(x) = \int_0^{3x} \sqrt{4 + t^2}\, dt$ and $g(x) = f(\sin x)$.

(a) Find $f'(x)$ and $g'(x)$.

(b) Write an equation for the line tangent to the graph of $y = g(x)$ at $x = \pi$.

(c) Write, but do not evaluate, an integral expression that represents the maximum value of $g$ on the interval $0 \le x \le \pi$. Justify your answer.

---

(a) $f'(x) = 3\sqrt{4 + (3x)^2}$

$g'(x) = f'(\sin x) \cdot \cos x$

$= 3\sqrt{4 + (3\sin x)^2} \cdot \cos x$

$4 : \begin{cases} 2 : f'(x) \\ 2 : g'(x) \end{cases}$

(b) $g(\pi) = 0$, $g'(\pi) = -6$
Tangent line: $y = -6(x - \pi)$

$2 : \begin{cases} 1 : g(\pi) \text{ or } g'(\pi) \\ 1 : \text{tangent line equation} \end{cases}$

(c) For $0 < x < \pi$, $g'(x) = 0$ only at $x = \frac{\pi}{2}$.

$g(0) = g(\pi) = 0$

$g\left(\frac{\pi}{2}\right) = \int_0^3 \sqrt{4 + t^2}\, dt > 0$

The maximum value of $g$ on $[0, \pi]$ is
$\int_0^3 \sqrt{4 + t^2}\, dt$.

$3 : \begin{cases} 1 : \text{sets } g'(x) = 0 \\ 1 : \text{justifies maximum at } \frac{\pi}{2} \\ 1 : \text{integral expression for } g\left(\frac{\pi}{2}\right) \end{cases}$



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### Question 6

Let $f$ be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let $g$ be the function given by $g(x) = f(f(x))$.

(a) Explain why there must be a value $c$ for $2 < c < 5$ such that $f'(c) = -1$.

(b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value $k$ for $2 < k < 5$ such that $g''(k) = 0$.

(c) Show that if $f''(x) = 0$ for all $x$, then the graph of $g$ does not have a point of inflection.

(d) Let $h(x) = f(x) - x$. Explain why there must be a value $r$ for $2 < r < 5$ such that $h(r) = 0$.

---

(a) The Mean Value Theorem guarantees that there is a value $c$, with $2 < c < 5$, so that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{2 - 5}{5 - 2} = -1.$$

$$2: \begin{cases} 1: \dfrac{f(5) - f(2)}{5 - 2} \\ 1: \text{conclusion, using MVT} \end{cases}$$

(b) $g'(x) = f'(f(x)) \cdot f'(x)$

$g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2)$

$g'(5) = f'(f(5)) \cdot f'(5) = f'(2) \cdot f'(5)$

Thus, $g'(2) = g'(5)$.

Since $f$ is twice-differentiable, $g'$ is differentiable everywhere, so the Mean Value Theorem applied to $g'$ on $[2, 5]$ guarantees there is a value $k$, with $2 < k < 5$, such that $g''(k) = \dfrac{g'(5) - g'(2)}{5 - 2} = 0$.

$$3: \begin{cases} 1: g'(x) \\ 1: g'(2) = f'(5) \cdot f'(2) = g'(5) \\ 1: \text{uses MVT with } g' \end{cases}$$

(c) $g''(x) = f''(f(x)) \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot f''(x)$

If $f''(x) = 0$ for all $x$, then

$g''(x) = 0 \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot 0 = 0$ for all $x$.

Thus, there is no $x$-value at which $g''(x)$ changes sign, so the graph of $g$ has no inflection points.

$$2: \begin{cases} 1: \text{considers } g'' \\ 1: g''(x) = 0 \text{ for all } x \end{cases}$$

OR

If $f''(x) = 0$ for all $x$, then $f$ is linear, so $g = f \circ f$ is linear and the graph of $g$ has no inflection points.

OR

$$2: \begin{cases} 1: f \text{ is linear} \\ 1: g \text{ is linear} \end{cases}$$

(d) Let $h(x) = f(x) - x$.

$h(2) = f(2) - 2 = 5 - 2 = 3$

$h(5) = f(5) - 5 = 2 - 5 = -3$

Since $h(2) > 0 > h(5)$, the Intermediate Value Theorem guarantees that there is a value $r$, with $2 < r < 5$, such that $h(r) = 0$.

$$2: \begin{cases} 1: h(2) \text{ and } h(5) \\ 1: \text{conclusion, using IVT} \end{cases}$$




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## Question 6

The twice-differentiable function $f$ is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2,\ f'(0) = -4, \text{ and } f''(0) = 3.$$

(a) The function $g$ is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where $a$ is a constant. Find $g'(0)$ and $g''(0)$ in terms of $a$. Show the work that leads to your answers.

(b) The function $h$ is given by $h(x) = \cos(kx)f(x)$ for all real numbers, where $k$ is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of $h$ at $x = 0$.

| | |
|---|---|
| (a) $g'(x) = ae^{ax} + f'(x)$<br>$g'(0) = a - 4$<br><br>$g''(x) = a^2e^{ax} + f''(x)$<br>$g''(0) = a^2 + 3$ | $4 : \begin{cases} 1 : g'(x) \\ 1 : g'(0) \\ 1 : g''(x) \\ 1 : g''(0) \end{cases}$ |
| (b) $h'(x) = f'(x)\cos(kx) - k\sin(kx)f(x)$<br>$h'(0) = f'(0)\cos(0) - k\sin(0)f(0) = f'(0) = -4$<br>$h(0) = \cos(0)f(0) = 2$<br>The equation of the tangent line is $y = -4x + 2$. | $5 : \begin{cases} 2 : h'(x) \\ 3 : \begin{cases} 1 : h'(0) \\ 1 : h(0) \\ 1 : \text{equation of tangent line} \end{cases} \end{cases}$ |

### Question 3

The figure above is the graph of a function of $x$, which models the height of a skateboard ramp. The function meets the following requirements.

(i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.

(ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.

(iii) Between $x = 0$ and $x = 4$, the function is increasing.


(a) Let $f(x) = ax^2$, where $a$ is a nonzero constant. Show that it is not possible to find a value for $a$ so that $f$ meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \dfrac{x^2}{16}$, where $c$ is a nonzero constant. Find the value of $c$ so that $g$ meets requirement (ii) above. Show the work that leads to your answer.

(c) Using the function $g$ and your value of $c$ from part (b), show that $g$ does not meet requirement (iii) above.

(d) Let $h(x) = \dfrac{x^n}{k}$, where $k$ is a nonzero constant and $n$ is a positive integer. Find the values of $k$ and $n$ so that $h$ meets requirement (ii) above. Show that $h$ also meets requirements (i) and (iii) above.

| | |
|---|---|
| (a) $f(4) = 1$ implies that $a = \frac{1}{16}$ and $f'(4) = 2a(4) = 1$ implies that $a = \frac{1}{8}$. Thus, $f$ cannot satisfy (ii). | $2 : \begin{cases} 1 : a = \frac{1}{16} \text{ or } a = \frac{1}{8} \\ 1 : \text{shows } a \text{ does not work} \end{cases}$ |
| (b) $g(4) = 64c - 1 = 1$ implies that $c = \frac{1}{32}$. When $c = \frac{1}{32}$, $g'(4) = 3c(4)^2 - \frac{2(4)}{16} = 3\left(\frac{1}{32}\right)(16) - \frac{1}{2} = 1$ | $1 :$ value of $c$ |
| (c) $g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = \frac{1}{32}x(3x - 4)$ <br> $g'(x) < 0$ for $0 < x < \frac{4}{3}$, so $g$ does not satisfy (iii). | $2 : \begin{cases} 1 : g'(x) \\ 1 : \text{explanation} \end{cases}$ |
| (d) $h(4) = \frac{4^n}{k} = 1$ implies that $4^n = k$. <br> $h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = \frac{n}{4} = 1$ gives $n = 4$ and $k = 4^4 = 256$. <br> $h(x) = \frac{x^4}{256} \Rightarrow h(0) = 0$. <br> $h'(x) = \frac{4x^3}{256} \Rightarrow h'(0) = 0$ and $h'(x) > 0$ for $0 < x < 4$. | $4 : \begin{cases} 1 : \frac{4^n}{k} = 1 \\ 1 : \frac{n4^{n-1}}{k} = 1 \\ 1 : \text{values for } k \text{ and } n \\ 1 : \text{verifications} \end{cases}$ |





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# BC Topics - Polar Coordinates

98 BC

19. The area of the region inside the polar curve $r = 4\sin\theta$ and outside the polar curve $r = 2$ is given by

(A) $\frac{1}{2}\int_0^{\pi}(4\sin\theta - 2)^2\,d\theta$ (B) $\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(4\sin\theta - 2)^2\,d\theta$ (C) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(4\sin\theta - 2)^2\,d\theta$

(D) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left(16\sin^2\theta - 4\right)d\theta$ (E) $\frac{1}{2}\int_0^{\pi}\left(16\sin^2\theta - 4\right)d\theta$

69 BC

9. The area of the closed region bounded by the polar graph of $r = \sqrt{3+\cos\theta}$ is given by the integral

(A) $\int_0^{2\pi}\sqrt{3+\cos\theta}\,d\theta$ (B) $\int_0^{\pi}\sqrt{3+\cos\theta}\,d\theta$ (C) $2\int_0^{\pi/2}(3+\cos\theta)\,d\theta$

(D) $\int_0^{\pi}(3+\cos\theta)\,d\theta$ (E) $2\int_0^{\pi/2}\sqrt{3+\cos\theta}\,d\theta$

97 BC

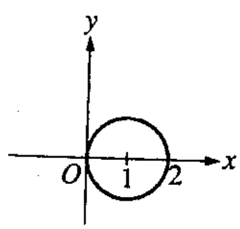
21. Which of the following is equal to the area of the region inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = \cos\theta$?

(A) $3\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$ (B) $3\int_0^{\pi}\cos^2\theta\,d\theta$ (C) $\frac{3}{2}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$ (D) $3\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$ (E) $3\int_0^{\pi}\cos\theta\,d\theta$

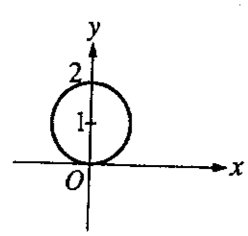
93 BC

5. Which of the following represents the graph of the polar curve $r = 2\sec\theta$?

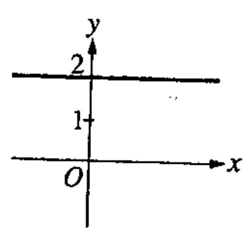
(A)



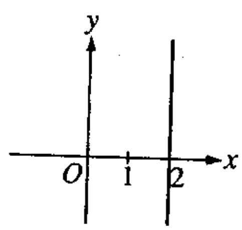

(B)



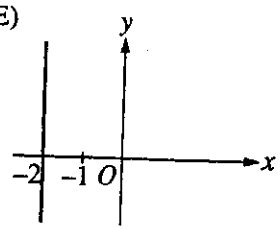

(C)




(D)




(E)

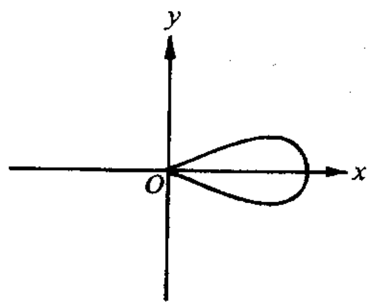


73 BC 40. The area of the region enclosed by the polar curve $r = 1 - \cos\theta$ is

(A) $\frac{3}{4}\pi$ (B) $\pi$ (C) $\frac{3}{2}\pi$ (D) $2\pi$ (E) $3\pi$

85 BC 24. The area of the region enclosed by the polar curve $r = \sin(2\theta)$ for $0 \le \theta \le \frac{\pi}{2}$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{\pi}{8}$ (E) $\frac{\pi}{4}$




88 BC 23. Which of the following gives the area of the region enclosed by the loop of the graph of the polar curve $r = 4\cos(3\theta)$ shown in the figure above?

(A) $16\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos(3\theta)\,d\theta$ (B) $8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos(3\theta)\,d\theta$ (C) $8\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2(3\theta)\,d\theta$

(D) $16\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\,d\theta$ (E) $8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\,d\theta$

# BC Topics - Parametric Equations

69 BC

1. The asymptotes of the graph of the parametric equations $x=\frac{1}{t}$, $y=\frac{t}{t+1}$ are

(A) $x=0,\ y=0$
(B) $x=0$ only
(C) $x=-1,\ y=0$
(D) $x=-1$ only
(E) $x=0,\ y=1$

97 BC

18. For what values of $t$ does the curve given by the parametric equations $x=t^3-t^2-1$ and $y=t^4+2t^2-8t$ have a vertical tangent?

(A) 0 only
(B) 1 only
(C) 0 and $\frac{2}{3}$ only
(D) 0, $\frac{2}{3}$, and 1
(E) No value

97 BC

15. The length of the path described by the parametric equations $x=\cos^3 t$ and $y=\sin^3 t$, for $0\le t\le\frac{\pi}{2}$, is given by

(A) $\int_0^{\frac{\pi}{2}} \sqrt{3\cos^2 t+3\sin^2 t}\, dt$

(B) $\int_0^{\frac{\pi}{2}} \sqrt{-3\cos^2 t\ \sin t+3\sin^2 t\ \cos t}\, dt$

(C) $\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t+9\sin^4 t}\, dt$

(D) $\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t\ \sin^2 t+9\sin^4 t\ \cos^2 t}\, dt$

(E) $\int_0^{\frac{\pi}{2}} \sqrt{\cos^6 t+\sin^6 t}\, dt$

88 BC

34. A curve in the plane is defined parametrically by the equations $x=t^3+t$ and $y=t^4+2t^2$. An equation of the line tangent to the curve at $t=1$ is

(A) $y=2x$
(B) $y=8x$
(C) $y=2x-1$
(D) $y=4x-5$
(E) $y=8x+13$

98
BC

21. The length of the path described by the parametric equations $x = \frac{1}{3}t^3$ and $y = \frac{1}{2}t^2$, where $0 \le t \le 1$, is given by

(A) $\int_0^1 \sqrt{t^2+1}\,dt$

(B) $\int_0^1 \sqrt{t^2+t}\,dt$

(C) $\int_0^1 \sqrt{t^4+t^2}\,dt$

(D) $\frac{1}{2}\int_0^1 \sqrt{4+t^4}\,dt$

(E) $\frac{1}{6}\int_0^1 t^2\sqrt{4t^2+9}\,dt$

98
BC

2. In the $xy$-plane, the graph of the parametric equations $x = 5t + 2$ and $y = 3t$, for $-3 \le t \le 3$, is a line segment with slope

(A) $\frac{3}{5}$ (B) $\frac{5}{3}$ (C) 3 (D) 5 (E) 13

# BC Topics - Vectors

88 BC 15. For any time $t \geq 0$, if the position of a particle in the $xy$-plane is given by $x = t^2 + 1$ and $y = \ln(2t+3)$, then the acceleration vector is

(A) $\left(2t, \frac{2}{(2t+3)}\right)$ (B) $\left(2t, \frac{-4}{(2t+3)^2}\right)$ (C) $\left(2, \frac{4}{(2t+3)^2}\right)$

(D) $\left(2, \frac{2}{(2t+3)^2}\right)$ (E) $\left(2, \frac{-4}{(2t+3)^2}\right)$

85 BC 4. A particle moves in the $xy$-plane so that at any time $t$ its coordinates are $x = t^2 - 1$ and $y = t^4 - 2t^3$. At $t = 1$, its acceleration vector is

(A) $(0,-1)$ (B) $(0,12)$ (C) $(2,-2)$ (D) $(2,0)$ (E) $(2,8)$

93 BC 28. If a particle moves in the $xy$-plane so that at time $t > 0$ its position vector is $\left(\ln(t^2+2t), 2t^2\right)$, then at time $t = 2$, its velocity vector is

(A) $\left(\frac{3}{4}, 8\right)$ (B) $\left(\frac{3}{4}, 4\right)$ (C) $\left(\frac{1}{8}, 8\right)$ (D) $\left(\frac{1}{8}, 4\right)$ (E) $\left(-\frac{5}{16}, 4\right)$

98 BC 77. If $f$ is a vector-valued function defined by $f(t) = \left(e^{-t}, \cos t\right)$, then $f''(t) =$

(A) $-e^{-t} + \sin t$ (B) $e^{-t} - \cos t$ (C) $\left(-e^{-t}, -\sin t\right)$

(D) $\left(e^{-t}, \cos t\right)$ (E) $\left(e^{-t}, -\cos t\right)$

# BC Topics - Sequences & Series

88 BC 13. $\sin(2x) =$

(A) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + \ldots$

(B) $2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots + \frac{(-1)^{n-1}(2x)^{2n-1}}{(2n-1)!} + \ldots$

(C) $-\frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots + \frac{(-1)^n(2x)^{2n}}{(2n)!} + \ldots$

(D) $\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$

(E) $2x + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \ldots + \frac{(2x)^{2n-1}}{(2n-1)!} + \ldots$

---

85 BC 14. Which of the following series are convergent?

I. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots$

II. $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$

III. $1 - \frac{1}{3} + \frac{1}{3^2} - \ldots + \frac{(-1)^{n+1}}{3^{n-1}} + \ldots$

(A) I only (B) III only (C) I and III only (D) II and III only (E) I, II, and III

---

85 BC 31. What are all values of $x$ for which the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges?

(A) $-1 \le x < 1$ (B) $-1 \le x \le 1$ (C) $0 < x < 2$ (D) $0 \le x < 2$ (E) $0 \le x \le 2$

---

88 BC 30. $\sum_{i=n}^{\infty} \left(\frac{1}{3}\right)^i =$

(A) $\frac{3}{2} - \left(\frac{1}{3}\right)^n$ (B) $\frac{3}{2}\left[1 - \left(\frac{1}{3}\right)^n\right]$ (C) $\frac{3}{2}\left(\frac{1}{3}\right)^n$

(D) $\frac{2}{3}\left(\frac{1}{3}\right)^n$ (E) $\frac{2}{3}\left(\frac{1}{3}\right)^{n+1}$

98 BC 18. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{n+2}$  II. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$  III. $\sum_{n=1}^{\infty} \frac{1}{n}$

(A) None
(B) II only
(C) III only
(D) I and II only
(E) I and III only

98 BC 14. What is the approximation of the value of $\sin 1$ obtained by using the fifth-degree Taylor polynomial about $x = 0$ for $\sin x$?

(A) $1 - \frac{1}{2} + \frac{1}{24}$

(B) $1 - \frac{1}{2} + \frac{1}{4}$

(C) $1 - \frac{1}{3} + \frac{1}{5}$

(D) $1 - \frac{1}{4} + \frac{1}{8}$

(E) $1 - \frac{1}{6} + \frac{1}{120}$

97 24. The expression $\frac{1}{50}\left(\sqrt{\frac{1}{50}} + \sqrt{\frac{2}{50}} + \sqrt{\frac{3}{50}} + \cdots + \sqrt{\frac{50}{50}}\right)$ is a Riemann sum approximation for

(A) $\int_0^1 \sqrt{\frac{x}{50}}\,dx$

(B) $\int_0^1 \sqrt{x}\,dx$

(C) $\frac{1}{50}\int_0^1 \sqrt{\frac{x}{50}}\,dx$

(D) $\frac{1}{50}\int_0^1 \sqrt{x}\,dx$

(E) $\frac{1}{50}\int_0^{50} \sqrt{x}\,dx$

300 300

97 BC

14. The sum of the infinite geometric series $\frac{3}{2}+\frac{9}{16}+\frac{27}{128}+\frac{81}{1,024}+\ldots$ is

(A) 1.60 (B) 2.35 (C) 2.40 (D) 2.45 (E) 2.50

97 BC

17. Let $f$ be the function given by $f(x)=\ln(3-x)$. The third-degree Taylor polynomial for $f$ about $x=2$ is

(A) $-(x-2)+\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

(B) $-(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

(C) $(x-2)+(x-2)^2+(x-2)^3$

(D) $(x-2)+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

(E) $(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

97 BC

20. What are all values of $x$ for which the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n\cdot 3^n}$ converges?

(A) $-3\le x\le 3$
(B) $-3<x<3$
(C) $-1<x\le 5$
(D) $-1\le x\le 5$
(E) $-1\le x<5$

97 BC

24. The Taylor series for $\sin x$ about $x=0$ is $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$. If $f$ is a function such that $f'(x)=\sin(x^2)$, then the coefficient of $x^7$ in the Taylor series for $f(x)$ about $x=0$ is

(A) $\frac{1}{7!}$ (B) $\frac{1}{7}$ (C) 0 (D) $-\frac{1}{42}$ (E) $-\frac{1}{7!}$

97 BC 25. The closed interval $[a,b]$ is partitioned into $n$ equal subintervals, each of width $\Delta x$, by the numbers $x_0, x_1, \ldots, x_n$ where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. What is $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{x_i}\,\Delta x$ ?

(A) $\frac{2}{3}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$

(B) $b^{\frac{3}{2}} - a^{\frac{3}{2}}$

(C) $\frac{3}{2}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$

(D) $b^{\frac{1}{2}} - a^{\frac{1}{2}}$

(E) $2\left(b^{\frac{1}{2}} - a^{\frac{1}{2}}\right)$

93 BC 16. Which of the following series diverge?

I. $\sum_{k=3}^{\infty} \frac{2}{k^2+1}$

II. $\sum_{k=1}^{\infty} \left(\frac{6}{7}\right)^k$

III. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k}$

~~(A) None (B) II only (C) III only (D) I and III (E) II and III~~

93 BC 27. The interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$ is

(A) $-3 < x \le 3$ (B) $-3 \le x \le 3$ (C) $-2 < x < 4$

(D) $-2 \le x < 4$ (E) $0 \le x \le 2$

93 BC 31. If $s_n = \left(\frac{(5+n)^{100}}{5^{n+1}}\right)\left(\frac{5^n}{(4+n)^{100}}\right)$, to what number does the sequence $\{s_n\}$ converge?

(A) $\frac{1}{5}$ (B) 1 (C) $\frac{5}{4}$ (D) $\left(\frac{5}{4}\right)^{100}$ (E) The sequence does not converge.

85 19. If $f(x_1)+f(x_2)=f(x_1+x_2)$ for all real numbers $x_1$ and $x_2$, which of the following could define $f$?

(A) $f(x)=x+1$ (B) $f(x)=2x$ (C) $f(x)=\frac{1}{x}$ (D) $f(x)=e^x$ (E) $f(x)=x^2$

85 BC 10. For $-1<x<1$ if $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{2n-1}$, then $f'(x)=$

(A) $\sum_{n=1}^{\infty}(-1)^{n+1}x^{2n-2}$

(B) $\sum_{n=1}^{\infty}(-1)^{n}x^{2n-2}$

(C) $\sum_{n=1}^{\infty}(-1)^{2n}x^{2n}$

(D) $\sum_{n=1}^{\infty}(-1)^{n}x^{2n}$

(E) $\sum_{n=1}^{\infty}(-1)^{n+1}x^{2n}$

93 BC 19. Which of the following series converge?

I. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ II. $\sum_{n=1}^{\infty}\frac{1}{n}$ III. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

98 BC 27. If $\sum_{n=0}^{\infty}a_nx^n$ is a Taylor series that converges to $f(x)$ for all real $x$, then $f'(1)=$

(A) 0 (B) $a_1$ (C) $\sum_{n=0}^{\infty}a_n$ (D) $\sum_{n=1}^{\infty}na_n$ (E) $\sum_{n=1}^{\infty}na_n^{\,n-1}$

98 BC 76. For what integer $k$, $k>1$, will both $\sum_{n=1}^{\infty}\frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty}\left(\frac{k}{4}\right)^n$ converge?

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

98 BC 22. If $\lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$ is finite, then which of the following must be true?

(A) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges

(B) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges

(C) $\sum_{n=1}^{\infty}\frac{1}{n^{p-2}}$ converges

(D) $\sum_{n=1}^{\infty}\frac{1}{n^{p-1}}$ converges

(E) $\sum_{n=1}^{\infty}\frac{1}{n^{p+1}}$ diverges

69 BC 30. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ is the Taylor series about zero for which of the following functions?

(A) $\sin x$ (B) $\cos x$ (C) $e^x$ (D) $e^{-x}$ (E) $\ln(1+x)$

69 BC 32. For what values of $x$ does the series $1+2^x+3^x+4^x+\cdots+n^x+\cdots$ converge?

(A) No values of $x$ (B) $x<-1$ (C) $x\geq -1$ (D) $x>-1$ (E) All values of $x$

85 BC 45. If $n$ is a positive integer, then $\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{3n}{n}\right)^2\right]$ can be expressed as

(A) $\int_0^1 \frac{1}{x^2}\,dx$ (B) $3\int_0^1\left(\frac{1}{x}\right)^2 dx$ (C) $\int_0^3\left(\frac{1}{x}\right)^2 dx$

(D) $\int_0^3 x^2\,dx$ (E) $3\int_0^3 x^2\,dx$

85 BC 42. The coefficient of $x^3$ in the Taylor series for $e^{3x}$ about $x=0$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{2}$ (E) $\frac{9}{2}$

98 BC 84. What are all values of $x$ for which the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$ converges?

(A) $-3 < x < -1$ (B) $-3 \le x < -1$ (C) $-3 \le x \le -1$ (D) $-1 \le x < 1$ (E) $-1 \le x \le 1$

98 BC 89. The graph of the function represented by the Maclaurin series $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + \frac{(-1)^n x^n}{n!} + \ldots$ intersects the graph of $y = x^3$ at $x =$

(A) 0.773 (B) 0.865 (C) 0.929 (D) 1.000 (E) 1.857

93 BC 43. The coefficient of $x^6$ in the Taylor series expansion about $x = 0$ for $f(x) = \sin\left(x^2\right)$ is

(A) $-\frac{1}{6}$ (B) 0 (C) $\frac{1}{120}$ (D) $\frac{1}{6}$ (E) 1

93 BC 45. If $f(x) = \sum_{k=1}^{\infty} \left(\sin^2 x\right)^k$, then $f(1)$ is

(A) 0.369 (B) 0.585 (C) 2.400 (D) 2.426 (E) 3.426

88 BC 38. What are all values of $x$ for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges?

(A) $-1 \le x \le 1$ (B) $-1 < x \le 1$ (C) $-1 \le x < 1$

(D) $-1 < x < 1$ (E) All real $x$

85 BC 41. $\lim_{n\to\infty} \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \ldots + \sqrt{\frac{n}{n}}\right] =$

(A) $\frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (B) $\int_0^1 \sqrt{x}\,dx$ (C) $\int_0^1 x\,dx$

(D) $\int_1^2 x\,dx$ (E) $2\int_1^2 x\sqrt{x}\,dx$

88 BC 44. Which of the following series converge?

I. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$

II. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{3}{2}\right)^n$

III. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) I, II, and III

---

69 BC 45. The complete interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(x+1)^k}{k^2}$ is

(A) $0 < x < 2$ (B) $0 \le x \le 2$ (C) $-2 < x \le 0$

(D) $-2 \le x < 0$ (E) $-2 \le x \le 0$

98 BC 83. The Taylor series for $\ln x$, centered at $x = 1$, is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$. Let $f$ be the function given by the sum of the first three nonzero terms of this series. The maximum value of $|\ln x - f(x)|$ for $0.3 \le x \le 1.7$ is

(A) 0.030 (B) 0.039 (C) 0.145 (D) 0.153 (E) 0.529

97 BC 76. Which of the following sequences converge?

I. $\left\{\frac{5n}{2n-1}\right\}$

II. $\left\{\frac{e^n}{n}\right\}$

III. $\left\{\frac{e^n}{1+e^n}\right\}$

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

# Integrals – BC

69 BC 22. If $f(x)=\int_0^x \frac{1}{\sqrt{t^3+2}}\,dt$, which of the following is FALSE?

(A) $f(0)=0$

(B) $f$ is continuous at $x$ for all $x \geq 0$.

(C) $f(1)>0$

(D) $f'(1)=\frac{1}{\sqrt{3}}$

(E) $f(-1)>0$

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88 BC 7. $\int_2^{+\infty} \frac{dx}{x^2}$ is

(A) $\frac{1}{2}$ (B) $\ln 2$ (C) $1$ (D) $2$ (E) nonexistent

---

58 BC 17. $\int_2^3 \frac{3}{(x-1)(x+2)}\,dx=$

(A) $-\frac{33}{20}$ (B) $-\frac{9}{20}$ (C) $\ln\left(\frac{5}{2}\right)$ (D) $\ln\left(\frac{8}{5}\right)$ (E) $\ln\left(\frac{2}{5}\right)$

---

88 26. $\int_0^{\frac{\pi}{2}} x\cos x\,dx=$

(A) $-\frac{\pi}{2}$ (B) $-1$ (C) $1-\frac{\pi}{2}$ (D) $1$ (E) $\frac{\pi}{2}-1$

69 BC 10. $\int_0^1 \frac{x^2}{x^2+1}\,dx=$

(A) $\frac{4-\pi}{4}$ (B) $\ln 2$ (C) $0$ (D) $\frac{1}{2}\ln 2$ (E) $\frac{4+\pi}{4}$

69 BC 29. $\int_0^1 \left(4-x^2\right)^{-\frac{3}{2}} dx =$

(A) $\frac{2-\sqrt{3}}{3}$ (B) $\frac{2\sqrt{3}-3}{4}$ (C) $\frac{\sqrt{3}}{12}$ (D) $\frac{\sqrt{3}}{3}$ (E) $\frac{\sqrt{3}}{2}$

93 BC 5. $\int_{-1}^{2} \frac{|x|}{x} dx$ is

(A) $-3$ (B) 1 (C) 2 (D) 3 (E) nonexistent

88 BC 31. $\int_0^2 \sqrt{4-x^2}\, dx =$

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\pi$ (D) $2\pi$ (E) $4\pi$

97 BC 1. $\int_0^1 \sqrt{x}\,(x+1)\, dx =$

(A) 0 (B) 1 (C) $\frac{16}{15}$ (D) $\frac{7}{5}$ (E) 2

93 BC 11. $\int_4^{\infty} \frac{-2x}{\sqrt[3]{9-x^2}} dx$ is

(A) $7^{\frac{2}{3}}$ (B) $\frac{3}{2}\left(7^{\frac{2}{3}}\right)$ (C) $9^{\frac{2}{3}}+7^{\frac{2}{3}}$ (D) $\frac{3}{2}\left(9^{\frac{2}{3}}+7^{\frac{2}{3}}\right)$ (E) nonexistent

97 BC 11. $\int_1^{\infty} \frac{x}{(1+x^2)^2} dx$ is

(A) $-\frac{1}{2}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$ (E) divergent

73 BC 36. $\int_0^1 \frac{x+1}{x^2+2x-3} dx$ is

(A) $-\ln\sqrt{3}$ (B) $-\frac{\ln\sqrt{3}}{2}$ (C) $\frac{1-\ln\sqrt{3}}{2}$ (D) $\ln\sqrt{3}$ (E) divergent

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97 BC 86. $\int \frac{dx}{(x-1)(x+3)} =$

(A) $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|+C$

(B) $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right|+C$

(C) $\frac{1}{2}\ln\left|(x-1)(x+3)\right|+C$

(D) $\frac{1}{2}\ln\left|\frac{2x+2}{(x-1)(x+3)}\right|+C$

(E) $\ln\left|(x-1)(x+3)\right|+C$

---

93 BC 29. $\int x\sec^2 x\,dx =$

(A) $x\tan x+C$ (B) $\frac{x^2}{2}\tan x+C$ (C) $\sec^2 x+2\sec^2 x\tan x+C$

(D) $x\tan x-\ln|\cos x|+C$ (E) $x\tan x+\ln|\cos x|+C$

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85 BC 28. An antiderivative of $f(x)=e^{x+e^x}$ is

(A) $\frac{e^{x+e^x}}{1+e^x}$ (B) $(1+e^x)e^{x+e^x}$ (C) $e^{1+e^x}$ (D) $e^{x+e^x}$ (E) $e^{e^x}$

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88 BC 16. $\int xe^{2x}\,dx =$

(A) $\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}+C$ (B) $\frac{xe^{2x}}{2}-\frac{e^{2x}}{2}+C$ (C) $\frac{xe^{2x}}{2}+\frac{e^{2x}}{4}+C$

(D) $\frac{xe^{2x}}{2}+\frac{e^{2x}}{2}+C$ (E) $\frac{x^2e^{2x}}{4}+C$

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85 BC 12. $\int \frac{dx}{(x-1)(x+2)} =$

(A) $\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$ (B) $\frac{1}{3}\ln\left|\frac{x+2}{x-1}\right|+C$ (C) $\frac{1}{3}\ln|(x-1)(x+2)|+C$

(D) $(\ln|x-1|)(\ln|x+2|)+C$ (E) $\ln\left|(x-1)(x+2)^2\right|+C$

98 BC 4. $\int \frac{1}{x^2-6x+8}\,dx =$

(A) $\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right|+C$

(B) $\frac{1}{2}\ln\left|\frac{x-2}{x-4}\right|+C$

(C) $\frac{1}{2}\ln\left|(x-2)(x-4)\right|+C$

(D) $\frac{1}{2}\ln\left|(x-4)(x+2)\right|+C$

(E) $\ln\left|(x-2)(x-4)\right|+C$

93 22. An antiderivative for $\frac{1}{x^2-2x+2}$ is

(A) $-(x^2-2x+2)^{-2}$

(B) $\ln(x^2-2x+2)$

(C) $\ln\left|\frac{x-2}{x+1}\right|$

(D) $\operatorname{arcsec}(x-1)$

(E) $\arctan(x-1)$

69 BC 42. If $\int x^2\cos x\,dx = f(x) - \int 2x\sin x\,dx$, then $f(x) =$

(A) $2\sin x + 2x\cos x + C$

(B) $x^2\sin x + C$

(C) $2x\cos x - x^2\sin x + C$

(D) $4\cos x - 2x\sin x + C$

(E) $\left(2-x^2\right)\cos x - 4\sin x + C$

98 BC 15. $\int x\cos x\,dx =$

(A) $x\sin x - \cos x + C$

(B) $x\sin x + \cos x + C$

(C) $-x\sin x + \cos x + C$

(D) $x\sin x + C$

(E) $\frac{1}{2}x^2\sin x + C$

97 BC 84. $\int x^2 \sin x\,dx =$

(A) $-x^2\cos x - 2x\sin x - 2\cos x + C$

(B) $-x^2\cos x + 2x\sin x - 2\cos x + C$

(C) $-x^2\cos x + 2x\sin x + 2\cos x + C$

(D) $-\frac{x^3}{3}\cos x + C$

(E) $2x\cos x + C$

93 BC 23. The length of the curve determined by the equations $x = t^2$ and $y = t$ from $t = 0$ to $t = 4$ is

(A) $\int_0^4 \sqrt{4t+1}\,dt$

(B) $2\int_0^4 \sqrt{t^2+1}\,dt$

(C) $\int_0^4 \sqrt{2t^2+1}\,dt$

(D) $\int_0^4 \sqrt{4t^2+1}\,dt$

(E) $2\pi\int_0^4 \sqrt{4t^2+1}\,dt$

88 BC 33. The length of the curve $y = x^3$ from $x = 0$ to $x = 2$ is given by

(A) $\int_0^2 \sqrt{1+x^6}\,dx$

(B) $\int_0^2 \sqrt{1+3x^2}\,dx$

(C) $\pi\int_0^2 \sqrt{1+9x^4}\,dx$

(D) $2\pi\int_0^2 \sqrt{1+9x^4}\,dx$

(E) $\int_0^2 \sqrt{1+9x^4}\,dx$

85 BC 23. $\lim_{h\to 0} \dfrac{\int_1^{1+h} \sqrt{x^5+8}\,dx}{h}$ is

(A) 0 (B) 1 (C) 3 (D) $2\sqrt{2}$ (E) nonexistent

93 BC 41. Let $f(x) = \int_{-2}^{x^2-3x} e^{t^2}\,dt$. At what value of $x$ is $f(x)$ a minimum?

(A) For no value of $x$ (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) 2 (E) 3

93 BC 32. If $\int_a^b f(x)dx = 5$ and $\int_a^b g(x)dx = -1$, which of the following must be true?

I. $f(x) > g(x)$ for $a \le x \le b$

II. $\int_a^b \left(f(x) + g(x)\right) dx = 4$

III. $\int_a^b \left(f(x)g(x)\right) dx = -5$

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

93 BC 35. If $F$ and $f$ are differentiable functions such that $F(x) = \int_0^x f(t)dt$, and if $F(a) = -2$ and $F(b) = -2$ where $a < b$, which of the following must be true?

(A) $f(x) = 0$ for some $x$ such that $a < x < b$.

(B) $f(x) > 0$ for all $x$ such that $a < x < b$.

(C) $f(x) < 0$ for all $x$ such that $a < x < b$.

(D) $F(x) \le 0$ for all $x$ such that $a < x < b$.

(E) $F(x) = 0$ for some $x$ such that $a < x < b$.

93 BC 3. If $p$ is a polynomial of degree $n$, $n > 0$, what is the degree of the polynomial $Q(x) = \int_0^x p(t)dt$?

(A) 0 (B) 1 (C) $n-1$ (D) $n$ (E) $n+1$

98 BC 82. If $f(x) = g(x) + 7$ for $3 \le x \le 5$, then $\int_3^5 [f(x) + g(x)]\,dx =$

(A) $2\int_3^5 g(x)\,dx + 7$

(B) $2\int_3^5 g(x)\,dx + 14$

(C) $2\int_3^5 g(x)\,dx + 28$

(D) $\int_3^5 g(x)\,dx + 7$

(E) $\int_3^5 g(x)\,dx + 14$

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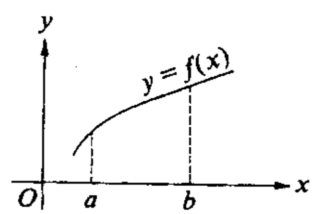
98 BC 28. $\lim_{x \to 1} \dfrac{\int_1^x e^{t^2}\,dt}{x^2 - 1}$ is

(A) 0 (B) 1 (C) $\frac{e}{2}$ (D) $e$ (E) nonexistent




85 BC 27. If $f$ is the continuous, strictly increasing function on the interval $a \le x \le b$ as shown above, which of the following must be true?

I. $\int_a^b f(x)\,dx < f(b)(b-a)$

II. $\int_a^b f(x)\,dx > f(a)(b-a)$

III. $\int_a^b f(x)\,dx = f(c)(b-a)$ for some number $c$ such that $a < c < b$

(A) I only (B) II only (C) III only (D) I and III only (E) I, II, and III

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97 BC 82. If $0 \le x \le 4$, of the following, which is the greatest value of $x$ such that $\int_0^x (t^2 - 2t)\,dt \ge \int_2^x t\,dt$?

(A) 1.35 (B) 1.38 (C) 1.41 (D) 1.48 (E) 1.59